Assignment 3B

Problem 1

(i) Consider

$$\psi: \mathbb{D} \to \mathbb{H}, \quad z \mapsto i\,\frac{1+z}{1-z},$$

$$\psi^{-1}: \mathbb{H} \to \mathbb{D}, \quad z \mapsto \frac{z-i}{z+i}.$$

Suppose that $\phi: \mathbb{H} \to \mathbb{H}$ is an automorphism of $\mathbb{H}$ satisfying

$$|\phi(z)| \to \infty \quad \text{as} \quad |z| \to \infty. \qquad (*)$$

Then $\tilde{\phi} := \psi^{-1} \circ \phi \circ \psi : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$, hence of the form

$$\tilde{\phi}(z) = e^{i\theta}\,\frac{z+a}{1+\bar{a}z}, \quad z \in \mathbb{D},$$

for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

Note that $\phi = \psi \circ \tilde{\phi} \circ \psi^{-1}$. We thus compute

$$\tilde{\phi}(\psi^{-1}(z)) = e^{i\theta}\,\frac{\frac{z-i}{z+i} + a}{1 + \bar{a}\,\frac{z-i}{z+i}}$$

$$= e^{i\theta}\,\frac{(z-i) + a(z+i)}{(z+i) + \bar{a}(z-i)}$$

$$= e^{i\theta}\,\frac{(1+a)z - (1-a)i}{(1+\bar{a})z + (1-\bar{a})i}$$

and

$$\phi(z) = \Psi\left(\tilde{\phi}\left(\Psi^{-1}(z)\right)\right) = i \, \frac{1 + e^{i\theta} \dfrac{(1+a)z - (1-a)i}{(1+\bar{a})z + (1-\bar{a})i}}{1 - e^{i\theta} \dfrac{(1+a)z - (1-a)i}{(1+\bar{a})z + (1-\bar{a})i}}$$

$$= i \, \frac{\left[(1+\bar{a})z + (1-\bar{a})i\right] + e^{i\theta}\left[(1+a)z - (1-a)i\right]}{\left[(1+\bar{a})z + (1-\bar{a})i\right] - e^{i\theta}\left[(1+a)z - (1-a)i\right]}$$

$$= i \, \frac{z\left[(1+\bar{a}) + e^{i\theta}(1+a)\right] + i\left[(1-\bar{a}) - e^{i\theta}(1-a)\right]}{z\left[(1+\bar{a}) - e^{i\theta}(1+a)\right] + i\left[(1-\bar{a}) + e^{i\theta}(1-a)\right]},$$

so that (x) implies that

$$(1+\bar{a}) - e^{i\theta}(1+a) = 0, \quad \text{i.e.} \quad e^{i\theta} = \frac{1+\bar{a}}{1-\bar{a}}.$$

In this case, $\phi(z) = \sigma z + \mu$, where

$$\sigma = \frac{(1+\bar{a}) + e^{i\theta}(1+a)}{(1-\bar{a}) + e^{i\theta}(1-a)} = \frac{(1+\bar{a})e^{-i\frac{\theta}{2}} + (1+a)e^{i\frac{\theta}{2}}}{(1-\bar{a})e^{-i\frac{\theta}{2}} + (1-a)e^{i\frac{\theta}{2}}},$$

$$\mu = i \, \frac{(1-\bar{a}) - e^{i\theta}(1-a)}{(1-\bar{a}) + e^{i\theta}(1-a)} = i \cdot \frac{(1-\bar{a})e^{-i\frac{\theta}{2}} - (1-a)e^{i\frac{\theta}{2}}}{(1-\bar{a})e^{-i\frac{\theta}{2}} + (1-a)e^{i\frac{\theta}{2}}}.$$

Thus, $\bar{\sigma} = \sigma$ and $\bar{\mu} = \mu$, i.e. $\sigma, \mu \in \mathbb{R}$.

$\sigma > 0$ follows, since $\phi : \mathbb{H} \to \mathbb{H}$ is non-constant.
($\sigma \leq 0$ would lead to a contradiction.)

(ii) If $g_i : \mathbb{H}\setminus K \to \mathbb{H}$, $i=1,2$ both satisfy the hydrodynamic normalization, then

$$\phi := g_2 \circ g_1^{-1} : \mathbb{H} \to \mathbb{H}$$

satisfies

$$\lim_{|z|\to\infty} \left(\phi(z) - z\right) = 0 \qquad (**)$$

Thus, in particular $|\phi(z)| \to \infty$ as $|z|\to\infty$, so that by (i)

$$\phi(z) = \sigma z + \mu \quad \text{for some } \sigma > 0, \mu \in \mathbb{R}.$$

(**) enforces then $\sigma = 1$ and $\mu = 0$, i.e. $\phi = id_{\mathbb{H}}$.

$$\Rightarrow \quad g_2 = g_1.$$

Problem 2

By the Riemann mapping theorem, there is a unique biholomorphic mapping

$$\tilde{\phi} : \tilde{G}_I \to \tilde{\mathbb{H}}_{(-1,1)} \quad \text{with} \quad \tilde{\phi}(x) = 0,\ \tilde{\phi}'(x) > 0.$$

Claim: $\tilde{\phi}$ is reflection symmetric, i.e.

$$\overline{\tilde{\phi}(z)} = \tilde{\phi}(\bar{z}) \quad \text{for all } z \in \tilde{G}_I.$$

Proof: Def. $\psi : \tilde{G}_I \to \tilde{\mathbb{H}}_{(-1,1)}$, $z \mapsto \overline{\tilde{\phi}(\bar{z})}$.

Since ψ is biholomorphic and satisfies $\psi(x) = 0$, $\psi'(x) > 0$, we must have $\psi = \tilde{\phi}$. $\square$

Thus: $\tilde{\phi}(I) \subseteq (-1,1)$ and $\tilde{\phi}^{-1}((-1,1)) \subseteq I$

$$\Longrightarrow \quad \tilde{\phi}(I) = (-1,1)$$

Claim: $\tilde{\phi}(G) \subseteq \mathbb{H}$ and $\tilde{\phi}^{-1}(\mathbb{H}) \subseteq G$

$$\Longrightarrow \quad \tilde{\phi}(G) = \mathbb{H}.$$

Proof: $\tilde{\phi}(G)$ is a domain, $\tilde{\phi}(G) \subseteq \widetilde{\mathbb{H}}_{(-1,1)}$, but

$$\tilde{\phi}(G) \cap (-1,1) = \emptyset \quad (\text{since } G \cap I = \emptyset)$$

$$\Longrightarrow \quad \tilde{\phi}(G) \subseteq \mathbb{H} \quad \text{or} \quad \tilde{\phi}(G) \subseteq -\mathbb{H}.$$

Now, note that

$$\lim_{\varepsilon \downarrow 0} \frac{\tilde{\phi}(x+i\varepsilon) - \overbrace{\tilde{\phi}(x)}^{=0}}{(x+i\varepsilon) - x} = \tilde{\phi}'(x) > 0$$

$$\Longrightarrow \quad \lim_{\varepsilon \downarrow 0} \underbrace{\mathrm{Re}\left(\frac{\tilde{\phi}(x+i\varepsilon)}{i\varepsilon}\right)}_{} = \tilde{\phi}'(x) > 0$$

$$= \frac{\mathrm{Im}(\tilde{\phi}(x+i\varepsilon))}{\varepsilon}$$

Thus, $\mathrm{Im}(\tilde{\phi}(x+i\varepsilon)) > 0$ for all small $\varepsilon > 0$

$$\Longrightarrow \quad \tilde{\phi}(G) \subseteq \mathbb{H}.$$

The same argument for $\tilde{\phi}^{-1}$ gives $\tilde{\phi}^{-1}(\mathbb{H}) \subseteq G$.

$\square$

Thus: $\tilde{\phi}$ restricts to a biholomorphic mapping

$$\phi := \phi|_G : G \to \mathbb{H}$$

with the required properties.

Uniqueness:

If $\psi : G \to \mathbb{H}$ is biholomorphic with these properties, it extends by reflection to a biholomorphic (!!) mapping $\tilde{\psi} : \tilde{G}_I \to \tilde{\mathbb{H}}_{(-1,1)}$ satisfying $\tilde{\psi}(x) = 0$ and $\tilde{\psi}'(x) > 0$.

By uniqueness in the Riemann mapping theorem

$$\tilde{\psi} = \tilde{\phi}$$

$$\Rightarrow \quad \psi = \tilde{\psi}|_G = \tilde{\phi}|_G = \phi \qquad \square$$