Assignment 4 A

Problem 1:

(i) For $g \in \Sigma_0 \subset \Sigma$, Grönwall's Area Theorem yields

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

Thus, in particular

$$n|b_n|^2 \leq 1 \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad |b_n| \leq \frac{1}{\sqrt{n}} \leq 1 \qquad \forall n \in \mathbb{N}. \qquad (*)$$

Write

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \qquad (b_0 = 0).$$

Then, for all $z \in \Delta_2$ (and even for $z \in \overline{\Delta}_2$)

$$|g(z) - z| = \left|\sum_{n=1}^{\infty} b_n z^{-n}\right|$$

$$\leq \sum_{n=1}^{\infty} |b_n| \frac{1}{|z|^n} \underset{|z|\geq 2}{\overset{(*)}{\leq}} \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

(ii) Fix $w \in \Delta_3$. Consider the holomorphic functions

$$f_1 : \mathbb{D}\setminus\{0\} \to \mathbb{C}, \quad z \mapsto z\left[g\left(\tfrac{1}{z}\right) - w\right] \qquad \text{and}$$

$$f_2 : \mathbb{D}\setminus\{0\} \to \mathbb{C}, \quad z \mapsto z\left[\tfrac{1}{z} - w\right],$$

which extend to holomorphic functions $f_1, f_2 : \mathbb{D} \to \mathbb{C}$.

Now, for $z \in \partial D(0,\frac{1}{2})$,

$$|f_1(z) - f_2(z)| = |z| \underbrace{\left| g\left(\tfrac{1}{z}\right) - \tfrac{1}{z} \right|}_{\leq 1 \text{ by (i), since } \left|\frac{1}{z}\right| = 2}$$

$$< |z| \underbrace{\left| \frac{1}{z} - w \right|}_{\geq |w| - \left|\frac{1}{z}\right| > 3-2 = 1}$$

$$\leq |f_2(z)|$$

$\Rightarrow$ f_1, f_2 have the same number of zeros (counted according to multiplicities) on $D(0,\frac{1}{2})$.

Now, $f_2(\frac{1}{w}) = 0$ with $\frac{1}{w} \in D(0,\frac{1}{2})$, implies

$$\exists\, z \in D(0,\tfrac{1}{2}) : \quad f_1(z) = 0$$

$$\Rightarrow \quad g\left(\tfrac{1}{z}\right) = w,$$

i.e., $\exists\, z \in \Delta_2 : \quad g(z) = w.$

Thus: $\quad \Delta_3 \subseteq g(\Delta_2)$

(iii) Using the expansion of g, we get that

$$g(z) - x = z - x + \sum_{n=1}^{\infty} b_n z^{-n}$$

$$\Rightarrow \quad \frac{g(z)-x}{z} = 1 - \frac{x}{z} + \sum_{n=1}^{\infty} b_n \frac{1}{z^{n+1}}$$

$$\Longrightarrow \left|\frac{g(z)-x}{z}\right| \geqslant 1 - \underbrace{\left|-\frac{x}{|z|} + \sum_{n=1}^{\infty} b_n \frac{1}{z^{n+1}}\right|}$$

$$\leqslant \frac{|x|}{|z|} + \sum_{n=1}^{\infty} |b_n| \frac{1}{|z|^{n+1}}$$

$x \in [-3,3]$

$(*)$

$$\leqslant \frac{3}{|z|} + \sum_{n=1}^{\infty} \frac{1}{|z|^{n+1}}$$

$|z| > 6$

$$< \frac{1}{2} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{6^{n+1}}}_{= \frac{1}{36} \cdot \frac{1}{1-\frac{1}{6}} = \frac{1}{30}} = \frac{8}{15}$$

$$> 1 - \frac{8}{15} = \frac{7}{15} \left(> \frac{3}{10} \right)$$

$$\Longrightarrow |g(z) - x| \geqslant \frac{7}{15}|z| \quad \text{for all } |z| > 6$$

Problem 2:

(i) Recall from the proof of Theorem 4.2 the construction of the mapping-out function g_K:
Consider $G := \{-\frac{1}{z} \mid z \in \mathbb{H} \setminus K\}$. If $K \subseteq \overline{D(0,r)}$, then $\mathbb{H} \cap \Delta_r \subseteq \mathbb{H} \setminus K$ and thus

$$G \supseteq \{-\tfrac{1}{z} \mid z \in \mathbb{H} \cap \Delta_r\} = D(0, \tfrac{1}{r}) \cap \mathbb{H}.$$

$$\Longrightarrow D(0, \tfrac{1}{r}) \subseteq G^* \quad \text{and} \quad I := (-\tfrac{1}{r}, \tfrac{1}{r}) \subseteq G^\circ$$

Consider (see Proposition 4.4) the biholomorphic mappings $\phi : G \to \mathbb{H}$ and

$$\tilde{\phi} : \tilde{G}_I \longrightarrow \tilde{\mathbb{H}}_{(-1,1)} \quad \text{with } \tilde{\phi}(0) = 0, \ \tilde{\phi}'(0) > 0.$$

Then $\tilde{\phi}(z) = b_1 z + b_2 z^2 + b_3 z^3 + O(|z|^4)$ and

$$g_K : \mathbb{H} \setminus K \to \mathbb{H}, \quad z \longmapsto -\frac{b_1}{\phi\left(-\frac{1}{z}\right)} - \frac{b_2}{b_1}.$$

Now, if we put

$$\tilde{g}_K : \Delta_r \to \mathbb{C}, \quad z \longmapsto -\frac{b_1}{\tilde{\phi}\left(-\frac{1}{z}\right)} - \frac{b_2}{b_1},$$

then

- $\tilde{g}_K$ is well-defined, since
  $$\left\{-\frac{1}{z} \;\middle|\; z \in \Delta_r\right\} = D\left(0, \frac{1}{r}\right) \setminus \{0\} \subset G^*,$$
- $\tilde{g}_K|_{\mathbb{H} \cap \Delta_r} = g_K|_{\mathbb{H} \cap \Delta_r}$, since
  $$\tilde{\phi}|_G = \phi,$$
- $\lim\limits_{|z| \to \infty} \left(\tilde{g}_K(z) - z\right) = 0$, since
  $$\tilde{g}_K(z) = z + \left(\left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1}\right)\frac{1}{z} + O(|z|^{-2}).$$

Clearly, $g : \Delta_1 \to \mathbb{C}, \ z \mapsto \frac{1}{r}\tilde{g}_K(rz)$

is holomorphic, schlicht, and satisfies

$$g(z) = z + \frac{1}{r^2}\left(\left(\frac{b_2}{b_1}\right)^2 - \frac{b_3}{b_1}\right) + O(|z|^{-2}).$$

Thus $g \in \Sigma_0$, so that by Problem 1 (ii)

$$\Delta_3 \subseteq g(\Delta_2) \quad \Rightarrow \quad \Delta_{3r} \subseteq \tilde{g}_K(\Delta_{2r})$$

and by Problem 1 (iii)

$$|g(z) - x| \geq \frac{3}{10}|z| \qquad \forall z \in \Delta_6, \; x \in [-3,3]$$

$$\Rightarrow \quad |\tilde{g}_K(rz) - rx| \geq \frac{3}{10}|rz| \quad \forall z \in \Delta_6, \; x \in [-3,3]$$

$$\Rightarrow \quad |\tilde{g}_K(z) - x| \geq \frac{3}{10}|z| \quad \forall z \in \Delta_{6r}, \; x \in [-3r, 3r]$$

(ii) It suffices to prove that

$$|g_K(z) - z| \leq 5r \qquad \forall z \in \mathbb{H} \setminus K \qquad (**)$$

for $K \subseteq \overline{D(0,r)}$. The general case follows by translating K.

- From Problem 1 (i), we infer by scaling like above that

$$|\tilde{g}_K(z) - z| \leq r \qquad \forall z \in \Delta_{2r}.$$

Thus, $(**)$ holds if $z \in \Delta_{2r} \cap \mathbb{H}$.

- If $z \in \mathbb{H}$ but $z \notin \Delta_{2r}$, i.e. $|z| \leq 2$, then

$$|g_K(z)| = |\tilde{g}_K(z)| \leq 3r$$

since otherwise $\tilde{g}_K(z) \in \Delta_{3r} \subseteq \tilde{g}_K(\Delta_{2r})$ by (i), contradicting the injectivity of $\tilde{g}_K$.

$\Rightarrow |g_K(z) - z| \leq |g_K(z)| + |z| \leq 3r + 2r = 5r.$