UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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High Dimensional Probability
with Applications to Big Data Sciences

Summer term 2020

Assignment 1: Appetizers
Due: Tuesday, May 19, 2020

Problem 1 (Integral Inequalities)**.** The aim of this exercise is to prove useful integral identities that are used to bound the expectation and p^{th} moments by tail probabilities.

1. Let X be a non-negative random variable. Show that

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)\, dt.$$

   The two sides of the identity are either finite or infinite simultaneously.

2. Show that for any random variable X (not necessarily non-negative):

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)\, dt - \int_{-\infty}^0 \mathbb{P}(X < t)\, dt.$$

3. Let X be a random variable and $p \in (0, \infty)$. Show that

$$\mathbb{E}|X|^p = \int_0^\infty p t^{p-1} \mathbb{P}(|X| > t)\ dt$$

   whenever the right-hand side exists.

Problem 2 (Chebyshev's Inequality)**.** Let X be a random variable with mean μ and variance σ^2. Prove Chebyshev's inequality: for any $t > 0$,

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}.$$

Please turn the page.

Problem 3 (Weak Law of Large Numbers)**.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ and variance σ^2. Show that for any $t > 0$,

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} \mu, \quad \text{in Probability.}$$

Problem 4 (Strong Law of Large Numbers)**.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ and variance σ^2. Then the Strong Law of Large Numbers (SLLN) states that: for any $t > 0$,

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} \mu \qquad a.s.$$

Prove the SLLN under the additional assumption that $\mathbb{E}|X_i|^4 < \infty$. *Hint: Show that* $\mathbb{E}(S_n - \mu)^4 = \mathcal{O}(n^{-2})$.

Problem 5 (Central Limit Theorem)**.** Prove the CLT: Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ and variance σ^2. Then

$$S_n := \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$