UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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High Dimensional Probability
with Applications to Big Data Sciences

Summer term 2020

Assignment 3
Due: Monday, June 8, 2020

Problem 11 (Exercises from lecture)**.**

1. Prove the equivalence properties of sub-Exponential random variables in Proposition 2.6.

2. Prove the centering property in Lemma 2.8.

Problem 12 (Hoeffding Inequality for bounded random variables)**.**
The aim of this exercise is to directly prove Hoeffding's inequality for bounded random variables. We start by proving Hoeffding's Lemma:
Let X be a centered random variable taking values in the interval $[a, b]$ a.s.. Then X is sub-Gaussian with $\mathbb{E}\exp(\lambda X) \leq \exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$.

1. Show that
$$\mathbb{E}\exp(\lambda X) \leq \frac{b}{b-a}e^{\lambda a} - \frac{a}{b-a}e^{\lambda b}.$$

2. Deduce Hoeffding's Lemma by analyzing the function $\Phi(u) = -\gamma u + \log(1 - \gamma + \gamma e^u)$ where $u = \lambda(b-a)$ and $\gamma = -a/(b-a)$.

Let $X_1, \ldots, X_n$ be independent centered random variables such that, for any $i \in \{1, \ldots, n\}$, $X_i \in [a_i, b_i]$ a.s..

3. Show that for all $t > 0$,
$$\mathbb{P}\Big(\big|\sum_{i=1}^{n} X_i\big| \geq t\Big) \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).$$

Problem 13 (Bennett Inequality)**.**
The aim of this exercise is to prove Bennett's inequality which, unlike Hoeffding, captures the variance. Let $b, \sigma \geq 0$ and X be a random variable such that

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 \leq \sigma^2 \quad \text{and} \quad |X| \leq b \text{ a.s..}$$

1. Prove that for all $\lambda > 0$,

$$\mathbb{E}\exp(\lambda X) \le \exp\Big(\frac{\sigma^2}{b^2}(e^{\lambda b} - \lambda b - 1)\Big).$$

Hint: Use Taylor expansion and the fact that $e^x \ge x + 1$.

Let $X_1, \dots, X_n$ be independent random variables such that, for any $i \in \{1, \dots, n\}$

$$\mathbb{E}X_i = 0, \quad \mathbb{E}X_i^2 \le \sigma_i^2 \quad \text{and} \quad |X_i| \le b \ \text{ a.s.}.$$

2. Show that for all $t > 0$,

$$\mathbb{P}\Big(\big|\sum_{i=1}^n X_i\big| \ge t\Big) \le 2\exp\Big(-\frac{\sigma^2}{b^2}\, h\Big(\frac{tb}{\sigma^2}\Big)\Big),$$

where $\sigma^2 = \sum_{i=1}^n \sigma_i^2$ and $h(x) = (1+x)\ln(1+x) - x$.

Problem 14 (Bernstein Inequality)**.**
The aim of this exercise is to prove Bernstein inequality which, similar to Bennett, captures the variance. Let $b, \sigma \ge 0$ and X be a random variable such that

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 \le \sigma^2 \quad \text{and} \quad \mathbb{E}X^k \le \frac{1}{2}k!\sigma^2 b^{k-2}, \quad \forall k \ge 3.$$

1. Prove that for all $\lambda \in [0, 1/b]$,

$$\mathbb{E}\exp(\lambda X) \le \exp\Big(\frac{\lambda^2\sigma^2}{2(1-\lambda b)}\Big).$$

Hint: Use Taylor expansion and the fact that $e^x \ge x + 1$.

Let $X_1, \dots, X_n$ be independent random variables such that, for any $i \in \{1, \dots, n\}$

$$\mathbb{E}X_i = 0, \quad \mathbb{E}X_i^2 \le \sigma_i^2 \quad \text{and} \quad \mathbb{E}X_i^k \le \frac{1}{2}k!\sigma_i^2 b^{k-2}, \quad \forall k \ge 3.$$

2. Show that for all $t > 0$,

$$\mathbb{P}\Big(\big|\sum_{i=1}^n X_i\big| \ge t\Big) \le 2\exp\Big(-\frac{t^2}{2(\sigma - tb)}\Big),$$

where $\sigma^2 = \sum_{i=1}^n \sigma_i^2$.

Conclusion: Bernstein inequality gives a weaker bound than Bennett but holds for a larger class of random variables. Indeed, instead of boundedness, we only require an appropriate control of the moments.