UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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High Dimensional Probability
with Applications to Big Data Sciences

Summer term 2020

Assignment 4: Part 2
Due: Thursday, July 2, 2020

Problem 16 (Quadratic form on an ϵ-Nets)**.**
Let $x \in \mathbb{R}^n$ and $\mathcal{N}$ be an ϵ-net of the sphere S^{n-1}. Show that

$$\sup_{y \in \mathcal{N}} \langle x, y \rangle \leq \|x\|_2 \leq \frac{1}{1-\epsilon} \sup_{y \in \mathcal{N}} \langle x, y \rangle.$$

Let A be an $m \times n$ matrix and $\epsilon \in [0, 1/2)$.

1. Show that, for any ϵ-net $\mathcal{N}$ of the sphere S^{n-1} and ϵ-net $\mathcal{M}$ of the sphere S^{m-1}, we have
$$\sup_{x \in \mathcal{N}, y \in \mathcal{M}} \langle Ax, y \rangle \leq \|A\| \leq \frac{1}{1-2\epsilon} \sup_{x \in \mathcal{N}, y \in \mathcal{M}} \langle Ax, y \rangle.$$

2. If $m = n$ and A is symmetric, show that
$$\sup_{x \in \mathcal{N}} \langle Ax, x \rangle \leq \|A\| \leq \frac{1}{1-2\epsilon} \sup_{x \in \mathcal{N}} \langle Ax, x \rangle.$$

Problem 17 (Norm of Gaussian matrices with variance profile)**.** Let $(g_{ij})_{1 \leq j \leq i \leq n}$ be a family of independent standard Gaussian random variables and consider the $n \times n$ symmetric matrix A whose entries are defined, for any $1 \leq j \leq i \leq n$, by $A_{ij} = b_{ij} g_{ij}$ with $b_{ij} \geq 0$. The aim of this exercise is to prove that with high probability

$$\|A\| \leq C\sigma\sqrt{n},$$

where $C > 0$ and $\sigma = \max_{j \leq i} b_{ij}$.

1. Prove that there exists a constant $c > 0$ such that
$$\mathbb{P}\Big(|\langle Ax, x \rangle| \geq u\Big) \leq 2 \exp\Big(-\frac{ct^2}{\sigma^2}\Big).$$

2. Deduce.