UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
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High Dimensional Probability
with Applications to Big Data Sciences

Summer term 2020

Assignment 5
Due: Wednesday, July 15, 2020

Problem 18 (Functions of matrices)**.**
Check the following properties for $n \times n$ symmetric matrices X and Y.

1. $\|X\| \le t$ if and only if $-tI \preceq X \preceq tI$.
2. If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, $X \preceq Y$, and X and Y commute, then $f(X) \preceq f(Y)$.
3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. If $f(x) \le g(x)$ for all $x \in \mathbb{R}$ satisfying $|x| \le K$, then $f(X) \le g(X)$ for all X satisfying $\|X\| \le K$.

Problem 19 (Matrix moment generating function)**.**
Let X be an $n \times n$ symmetric random matrix with mean 0 such that $\|X\| \le 1$ almost surely. Prove the following bound:

$$\mathbb{E} \exp(\lambda X) \preceq \exp\left(g(\lambda)\mathbb{E}X^2\right) \quad \text{where} \quad g(\lambda) = \frac{\lambda^2/2}{1-\lambda/3},$$

provided that $0 \le \lambda < 3$.
Hint: Prove that $e^z \le 1 + z + \frac{1}{1-z/3} \cdot \frac{z^2}{2}$ *for* $|z| < 3$.

Problem 20 (Norm of the sum of independent random matrices)**.**
Let $X_1, \dots, X_N$ be independent, mean zero, $d \times d$ symmetric random matrices, such that $\|X_i\| \le K$ almost surely for all i. Deduce from Matrix Bernstein's inequality that

$$\mathbb{E}\Big\| \sum_{i=1}^{N} X_i \Big\| \lesssim \Big\| \sum_{i=1}^{N} \mathbb{E}X_i^2 \Big\|^{1/2} \sqrt{\log d} + K \log d.$$

Problem 21 (Matrix Hoeffding Inequality)**.**
Let $\varepsilon_1, \dots, \varepsilon_N$ be i.i.d. standard Gaussian random variables Let $A_1, \dots, A_N$ be $d \times d$ symmetric deterministic matrices. Prove that for any $t > 0$,

$$\mathbb{P}\Big(\lambda_{\max}\big(\sum_{i=1}^{N} \varepsilon_i A_i\big) > t\Big) \le d \exp\Big(-\frac{t^2}{2\sigma^2}\Big),$$

where $\sigma^2 = \| \sum_{i=1}^{N} A_i^2 \|$.

Problem 22 (Covariance estimation for general distributions)**.**
Let X be a mean-zero random vector in $\mathbb{R}^n$ with covariance matrix $\Sigma = \mathbb{E}XX^T$. Let $m \in \mathbb{N}$. Let $X_1, \dots, X_m$ be independent copies of X and consider the sample covariance matrix

$$\Sigma_m = \frac{1}{m} \sum_{i=1}^{m} X_i X_i^T.$$

1. Prove that if, for some $K > 1$, $\|X\|_2^2 \le K^2 \mathbb{E}\|X\|_2^2$ almost surely, then there exists a positive constant C such that

$$\mathbb{E}\|\Sigma_m - \Sigma\| \le C\Big(\sqrt{\frac{K^2 n \log n}{m}} + \frac{K^2 n \log n}{m}\Big)\|\Sigma\|.$$

2. Comment and compare with the sub-Gaussian case.