(over $\mathbb{C}$)	vector space	algebra	(unital)	$*$–algebra	pre-C^*–algebra
algebraic	$\mathcal{A}$	product $(a,b) \mapsto ab$ bilinear associative	$\exists(!)\mathbf{1}$ such that $\mathbf{1}a = a = a\mathbf{1}$ $(\mathbf{1}^* = \mathbf{1}^*\mathbf{1} = \mathbf{1} \leftarrow)$	involution $*\colon a \mapsto a^*$ self-inverse anti-linear anti-multiplicative	[Tentative: $\mathcal{A} \subset \mathcal{L}^a(H)$ (H a pre-Hilbert space)]
normed	$\lVert\bullet\rVert$	submultiplicative $\lVert ab\rVert \le \lVert a\rVert\,\lVert b\rVert$	$\lVert\mathbf{1}\rVert = 1$ ($\leftarrow$ only gives $\lVert\mathbf{1}\rVert \ge 1$)	isometric $\lVert a^*\rVert = \lVert a\rVert$	C^*–norm: $\lVert a^*a\rVert = \lVert a\rVert^2$ ($\Rightarrow \lVert a^*\rVert \ge \lVert a\rVert\ \forall a \Rightarrow \lVert a^*\rVert = \lVert a\rVert$, similarly, $0 \ne \lVert\mathbf{1}\rVert = \lVert\mathbf{1}^*\mathbf{1}\rVert = \lVert\mathbf{1}\rVert^2$)
Banach	complete	Banach algebra	unital Banach algebra	Banach $*$–algebra	C^*–algebra

Fundamental structure theorems on C^*–algebras:

- Every commutative C^*–algebra $\mathcal{A}$ is isomorphic to $C_0(\Omega)$ for a (unique) locally compact space Ω.

  [$\Omega = \{\tau\colon \mathcal{A} \xrightarrow[\text{multiplicative}]{\text{linear}} \mathbb{C}\}$ is a subset of the dual of $\mathcal{A}$, which is locally compact with respect to the weak* topology. The isomorphism $\mathcal{A} \to C_0(\Omega)$ is given by $a \mapsto (\widehat{a}\colon \tau \mapsto \tau(a))$.]

  [$\mathcal{A}$ is unital if and only if Ω is compact. Many proofs require that $\mathcal{A}$ is *unitalized* $\mathcal{A} \subset \widetilde{\mathcal{A}} := \mathcal{A} \oplus \widetilde{\mathbf{1}}\mathbb{C}$.]

- Every C^*–algebra $\mathcal{A}$ is isomorphic to a norm-closed $*$–subalgebra of some $\mathcal{B}(H)$.

  [For every element $a \in \mathcal{A}$ there exists a *state* φ such that $\varphi(a^*a) = \lVert a\rVert^2$, so that the *GNS-representation* for φ sends a to an operator with the same norm. It follows that the direct sum over all GNS-representations of states (called ***universal representation***) is *faithful*.]

Note! The first structure theorem is much more useful and fundamental than the second, because the isomorphism is **onto** a canonical object. (For every function $f \in C_0(\Omega)$ there is a unique element $a \in \mathcal{A}$ such that $\widehat{a} = f$. So, whatever you can do with function, you can do also in your commutative C^*–algebra, and theorems about the algebra structure of $C_0(\Omega)$ become theorems about $\mathcal{A}$.) It becomes most important when applied to suitable commutative subalgebras of general C^*–algebras. For instance, if a is a ***normal*** ($a^*a = aa^*$) element of $\mathcal{A} \ni \mathbf{1}$, then the C^*–subalgebra $C^*(a, \mathbf{1})$ of $\mathcal{A}$ ***generated*** by a and $\mathbf{1}$ is isomorphic to $C(\sigma(a))$ where $\sigma(a)$ is the *spectrum* of a (see below). So for every continuous function on the (compact) subset $\sigma(a)$ of $\mathbb{C}$, it makes sense to speak about $f(a)$ (the unique element in $C^*(a, \mathbf{1}) \subset \mathcal{A}$ such that $\widehat{f(a)} = f$). The isomorphism $C(\sigma(a)) \to C^*(a, \mathbf{1}) \subset \mathcal{A}$ is known as (***continuous***) ***spectral calculus*** at a. Frequently, also if a is not normal, one gets important information about a by looking at the normal element a^*a of $\mathcal{A}$.

The isomorphism in the second structure theorem is (usually) **not** onto the object $\mathcal{B}(H)$. That leaves the (sometimes quite tricky) problem to decide if an element of $\mathcal{B}(H)$ belongs to the C^*–algebra or not. (Operations you can do in $\mathcal{B}(H)$, like, for instance, taking supremum of two (or more) positive elements, may lead you out of the C^*–algebra.) Also, the object $\mathcal{B}(H)$ is far from being canonical or "nice". For instance, the *universal representation* constructed in the proof is nonseparable, whenever $\dim \mathcal{A} \geq 2$.

A useful tool. In a unital C^*–algebra (Banach algebra) $\mathcal{A}$, the ***Neumann series*** $\sum_{n=0}^{\infty} a^n$ exists whenever $\|a\| < 1$ and, of course, equals $(\mathbf{1} - a)^{-1}$. (Exercise: Use this to show that the set $inv(\mathcal{A})$ of invertible elements is open in $\mathcal{A}$.)

Another useful tool. If $\mathcal{A}$ is a C^*–algebra, we denote by $\widetilde{\mathcal{A}} := \mathcal{A} \oplus \widetilde{\mathbf{1}}\mathbb{C}$ its ***unitalization***. $\widetilde{\mathcal{A}}$ has a (unique C^*–)norm (making it a C^*–algebra). However, the way to obtain this norm depends on whether $\mathcal{A}$ is unital or not. If $\mathcal{A}$ is nonunital, we take the operator norm in the space $\mathcal{B}(\mathcal{A})$ of bounded linear maps on $\mathcal{A}$, by letting act the elements of $\widetilde{\mathcal{A}}$ on the *ideal* $\mathcal{A}$ via left multiplication. If $\mathcal{A}$ has already a unit $\mathbf{1}$, $\widetilde{\mathcal{A}}$ is isomorphic to the ***$*$–algebraic direct sum*** (that is, all operations component-wise) $\mathbb{C} \oplus \mathcal{A}$ (identifying $\mathbb{C}(\widetilde{\mathbf{1}} - \mathbf{1})$ with $\mathbb{C}$). The direct sum of a family of (pre-)C^*–algebras, is normed by the supremum of the norms of all components.

Spectrum. The spectrum generalizes the set of eigen-values of a matrix. The ***spectrum*** of an element a in a unital C^*–algebra (unital Banach algebra) is the compact (why?) subset

$$\sigma(a) \;:=\; \{\lambda \in \mathbb{C} \colon a - \lambda\mathbf{1} \notin inv(\mathcal{A})\}$$

of $\mathbb{C}$. (Exercise: Show that the spectrum of a matrix consist of the eigen-values of the matrix.) If $\mathcal{A}$ is nonunital, then $\sigma(a) := \widetilde{\sigma}(a)$, the spectrum of $a \in \widetilde{\mathcal{A}}$. (Exercise: If $\mathcal{A}$ is unital, show that $\widetilde{\sigma}(a) = \sigma(a) \cup \{0\}$. Exercise: Show that if a is invertible, then $a^{-1} \in C^*(a)$. (Hint: Examine first a^*a.) Consequently, the spectrum of a in $\mathcal{A} \ni \mathbf{1}_{\mathcal{A}}$ coincides with spectrum of a in any C^*–subalgebra $\mathcal{B} \ni \mathbf{1}_{\mathcal{A}}$ containing a.) Exercise: For every unital $*$–homomorphism φ, $\sigma(a) \supset \sigma(\varphi(a))$. Consequently, for every homomorphism, $\sigma(a) \cup \{0\} \supset \sigma(\varphi(a)) \cup \{0\}$.

Positive elements. An element a in a C^*–algebra $\mathcal{A}$ is ***positive*** if it is ***self-adjoint*** ($a^* = a$) and if $\sigma(a) \subset \mathbb{R}_+ := [0, \infty)$.

Equivalently: a is positive if $a = b^*b$ for some $b \in \mathcal{A}$. (Supplement: Every positive element a has a unique positive square root $\sqrt{a} \geq 0$, $\sqrt{a}^2 = a$. This follows, for instance, by spectral calculus. One also may take a sequence of polynomials $p_n(\lambda)$ that converge to $\sqrt{\lambda}$ uniformly on $0 \leq \lambda \leq \|a\|$ and show that the corresponding polynomials $p_n(a)$ converge in norm to something fulfilling the properties of $\sqrt{a}$.)

Equivalently: a is positive if $\varphi(a) \geq 0$ for all *positive* linear functionals φ on $\mathcal{A}$. (A linear functional φ on $\mathcal{A}$ is ***positive*** if $\varphi(b^*b) \geq 0$ for all $b \in \mathcal{A}$.)

Different definitions facilitate different purposes. For instance, only the third property makes it easy to show that $a \geq 0, b \geq 0 \Rightarrow a + b \geq 0$. We will see in the lecture a lemma that allows to conclude back from the third property to the others. The fact that b^*b **is** positive, has been an additional axiom for quite a while before Gelfand and Naimark were able to prove it from the other axioms of C^*–algebra.

If $a = a^*$, then $\sigma(a) \subset \mathbb{R}$. (The converse: A normal(!) element is self-adjoint if the spectrum is real.) Some properties we will use frequently:

- By spectral calculus, it follows that $a = a^*$ can be decomposed into $a = a_+ - a_-$ where $a_\pm \geq 0$ and $a_+a_- = 0$. It is a bit more difficult to show that such a decomposition into positive and negative part of a self-adjoint element ist unique. Since every $a \in \mathcal{A}$ can be decomposed into its ***real part*** $\frac{a+a^*}{2}$ and its ***imaginary part*** $\frac{a-a^*}{2i}$, which are self-adjoint, every a is a linear combination of four positive elements,

- If $a \geq 0$ then $b^*ab \geq 0$. Since $\|a\| \mathbf{1} \geq a = a^*$ (spectral calculus), we find $\|a\| b^*b \geq b^*ab$.

- $u \in \mathcal{A}$ is ***unitary*** if $u^*u = \mathbf{1} = uu^*$. If $a = a^*$ and $\|a\| \leq 1$, then $a + i\sqrt{\mathbf{1} - a^2}$ is a unitary. It follows that every a in a unital C^*–algebra is a linear combination of four unitaries. (If $\mathcal{A}$ is nonunital, unitalize, and observe that every unitary u in $\widetilde{\mathcal{A}}$ can be written as $u = \lambda(v + \mathbf{1})$ with $v \in \mathcal{A}$, fulfilling $v^*v + v + v^* = 0 = vv^* + v + v^*$. Such elements are called ***quasi-unitaries***; we conclude that every element in a non-unital C^*–algebra is a linear combination of four quasi-unitaries.) Unitaries have spectrum contained in the ***torus*** $\Pi := \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$ and the spectrum of quasi unitaries is contained in $\Pi - 1$. Also the converse is true: A normal element with spectrum in Π (in $\Pi - 1$) is a unitary (a quasi-unitary).

Spectral radius: The ***spectral radius*** of a is $r(a) := \sup|\sigma(a)|$. Theorem: If a is normal, then $r(a) = \|a\|$; in particular, $r(a^*a) = \|a\|^2$ for all a. Corollary: Every ($*$–)homomorphism ϑ between C^*–algebras is a ***contraction*** ($\|\vartheta(a)\| \leq \|a\|$). Corollary: An isomorphism between C^*–algebras is norm-preserving. In other words, on a $*$–algebra there exists at most one norm making it into a C^*–algebra. (However, there may exist different norms turning it into a pre-C^*–algebra.)

Some puzzles:

1. In general, we say an element a in a pre-C^*–algebra is positive, if a is positive in the completion $\overline{\mathcal{A}}$ of $\mathcal{A}$. Let $\mathcal{A}$ be the unital $*$–algebra generated by one self-adjoint indeterminate x. (That is, the complex polynomials in x.) Find two C^*–norms on $\mathcal{A}$, the first such that the element $x \in \mathcal{A}$ is positive, the second such that the element $-x \in \mathcal{A}$ is positive.)

   (Note: An element a in a (pre-)C^*–algebra that is positive **and** negative is 0. So, the fact that you may find a solution to the present exercise is, indeed, puzzling. It just shows that the spectrum calculated in a $*$–algebra does not always tell you essential things. It is the C^*–norm that tells you how to complete, and the completion may contain very essential new elements as compared with the original algebra.)

2. A ***projection*** in a (pre-)C^*–algebra is an element p satisfying $p^*p = p$ (so that $p^* = (p^*p)^* = p^*p = p$). Let p and q be projections. Then the following properties are equivalent.

   (a) $p \geq q$

   (b) $p - q$ is a projection.

   (c) $pq = q$ (or, equivalently, $qp = q$).

   (d) $qpq = q$.

   (e) $pqp = q$.

3. Let a, a', b be elements in a (pre-)C^*–algebra.

   (a) $ab = 0$ if and only if $abb^* = 0$.

   (b) Let $a \geq 0$, and $a' \geq 0$. Then $(a + a')b = 0$ if and only if $ab = 0$ and $a'b = 0$.

4. If $0 \leq b_n \to b$, then $0 \leq b$.

Quotients. If $\mathcal{A}$ is a C^*–algebra and $\mathcal{I}$ a closed ideal in $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ with norm $\|a + \mathcal{I}\| := \inf_{i \in \mathcal{I}} \|a + i\|$ is a C^*–algebra, too. Corollary: If $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a ($*$–)homomorphism, then $\varphi(\mathcal{B})$ is a C^*–subalgebra of $\mathcal{B}$. (Proof: Only closedness of $\varphi(\mathcal{B})$ is a question. Since φ is bounded, the ideal $\mathcal{I} := \ker\varphi$ is closed. Since $\mathcal{A}/\mathcal{I}$ is a C^*–algebra, the injective homomorphism $\hat{\varphi}\colon a + \mathcal{I} \mapsto \varphi(a)$ is isometric. Therefore, its image $\hat{\varphi}(\mathcal{A}/\mathcal{I}) = \varphi(\mathcal{B})$ is closed.)

Convention: If $R \times S \ni (r, s) \mapsto rs \in T$ is a map (called something like "product") and $R' \subset R, S' \subset S$, then **for us(!)**,

$$R'S' \;:=\; \{rs\colon r \in R', s \in S'\} \;\subset\; T,$$

the set of all products from R' and S'. Even if T is a vector space, we do **not** mean $\operatorname{span} R'S'$, and even if T is a normed space we do **not** mean $\overline{\operatorname{span}}\, R'S'$.