Problems: “Hilbert modules and their applications”
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No. 1 (discussion October 28)

1. (a) Show: In a Pre-Hilbert module, $\langle y, x\rangle = \langle y, x'\rangle$ for all y implies $x = x'$ gilt.

   (b) Show: In a pre-Hilbert module over a unital C^*–algebra we have $x\mathbf{1} = x$.

   [Supplement (may replace (1b)): Is $(u_\lambda)_{\lambda\in\Lambda}$ an ***approximate unit*** for the (not necessarily unital) C^*–algebra $\mathcal{B}$ (that is, for us, $\lim_\lambda bu_\lambda = b = \lim_\lambda u_\lambda b$ for every $b \in \mathcal{B}$), then $\lim_\lambda xu_\lambda = x$ for all x in a pre-Hilbert $\mathcal{B}$–module.]

2. Let $a\colon E \to F$ be an ***adjointable*** between pre-Hilbert $\mathcal{B}$–moduls, that is, there exists a map $a^*\colon F \to E$, the ***adjoint*** of a, such that $\langle ax, y\rangle = \langle x, a^*y\rangle$ for all $x \in E$ and $y \in F$. Show:

   (a) a is linear.

   (b) a^* is unique.

   (c) a^* is adjointable.

   (d) a is ***closable***, that is, for every sequence $(x_n)_{n\in\mathbb{N}}$ in E that converges to $x \in E$, we have $\lim_{n\to\infty} ax_n = y \Rightarrow y = ax$.

   [Supplement (may replace (2d)): To what extent the statements (2a)–(2c) remain true for semi-Hilbert modules?]

3. A ***projection*** on a pre-Hilbert module E is a map $p\colon E \to E$ such that $\langle px, py\rangle = \langle x, py\rangle$ for all $x, y \in E$. Show: p is a self-adjoint idempotent. If $p \neq 0$, then $\|p\| = 1$.

   [Supplement: What changes if we require only $\langle px, px\rangle = \langle x, px\rangle$?]

4. Show that the ***transposition*** $A = (a_{i,j})_{i,j} \mapsto A^t = (a_{j,i})_{i,j}$ on M_2 $(= M_2(\mathbb{C}))$ is ***positive*** ($A \geq 0 \Rightarrow A^t \geq 0$) but not 2–***positive*** ($\begin{pmatrix} A & B \\ C & D\end{pmatrix} \mapsto \begin{pmatrix} A^t & B^t \\ C^t & D^t\end{pmatrix}$ is not positive on $M_2(M_2) = M_4$).

WE 1. The inner product of a Hilbert module determines the module action uniquely. At some point in the lecture we even will show that a vector space E with a sesquilinear map $(\bullet, \bullet)\colon E \times E \to \mathcal{B}$, can be embedded into a Hilbert $\mathcal{B}$–module with inner product $\langle\bullet, \bullet\rangle$ such that $\langle x, y\rangle = (x, y)$ for all $x, y \in E$ (uniquely, if the Hilbert $\mathcal{B}$–module is generated as a Hilbert module by its subspace E) if and only if for each finite choice $x_i \in E$, $i = 1, \ldots, n$ the matrix $((x_i, x_j))_{i,j}$ is positive in $M_n(\mathcal{B})$.

Show that to satisfy that condition, it is not sufficient that $(\bullet, \bullet)$ is ***positive*** ($(x, x) \geq 0$) and ***definite*** ($(x, x) = 0 \Rightarrow x = 0$). In other words, find E with positive and definite $(\bullet, \bullet)$ and elements $x_i \in E$, $i = 1, \ldots, n$, such that the matrix $((x_i, x_j))_{i,j}$ is not positive in $M_n(\mathcal{B})$.