Problems: "Hilbert modules and their applications"

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No. 2 (discussion November 3)

1. Let x be in a Hilbert $\mathcal{B}$–module E and denote $C := C^*(|x|)$, the C^*–subalgebra of $\mathcal{B}$ generated by $|x| := \sqrt{\langle x, x\rangle} \in \mathcal{B}$. Show:

   (a) $x \mapsto |x|$ extends to a (unique) unitary between the Hilbert C–modules $F := \overline{xC}$ and C.

   (b) For every $\alpha \in (0, 1)$ there exists x_α (unique in F) such that $x = x\,|x|^\alpha$.

2. Let $E := E_1 \oplus E_2$ (E_i pre-Hilbert $\mathcal{B}$–modules), which we denote frequently also as $E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$. Show:

   (a)
$$\mathcal{B}^r\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \mathcal{B}^r(E_1) & \mathcal{B}^r(E_2, E_1) \\ \mathcal{B}^r(E_1, E_2) & \mathcal{B}^r(E_2) \end{pmatrix}.$$

   (b)
$$\mathcal{B}^a\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \mathcal{B}^a(E_1) & \mathcal{B}^a(E_2, E_1) \\ \mathcal{B}^a(E_1, E_2) & \mathcal{B}^a(E_2) \end{pmatrix},$$

   that is, $a \in \mathcal{B}^r(E)$ is adjointable if and only if each of its components in $\mathcal{B}^r(E_j, E_i)$ is adjointable.

3. Recall that for a subset S of a pre-Hilbert module E we define its ***orthogonal complement*** as $S^\perp := \{x \in E : \langle S, x\rangle = \{0\}\}$.

   (a) Show: If T is another subset of E, then $S \subset T^\perp$ implies $S^{\perp\perp} \subset T^\perp$.

   (b) Suppose $a \in \mathcal{B}^a(E, E')$ and $aS = \{0\}$. Show: $a(S^{\perp\perp}) = \{0\}$, too.

4. Show: If E is ***full*** (that is, if the ***range ideal*** $\mathcal{B}_E := \overline{\operatorname{span}}\langle E, E\rangle$ of E is all of $\mathcal{B}$), then $\sup_{\|x\|\le 1} \|xb\| = \|b\|$ for all $b \in \mathcal{B}$. (**Attention:** There's a trick here**!**)

WE 2. Let V be a normed space, let S be a subset of V, and let $p \in \mathcal{L}(V)$ be an idempotent (that is, $p^2 = p$). Examine to what extent additional properties of V, S, p (relative to each other or individual) determine the answer to the question if

$$\overline{\operatorname{span}}(pS) = p(\overline{\operatorname{span}}\, S).$$

(Here, the bar in $\overline{\operatorname{span}}$ means closure, not completion.)