Problems: "Hilbert modules and their applications"
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No. 3 (discussion November 10)

1. Let E be a Hilbert module and let F be a closed submodule of E. Show for the submodule F^∞ of E^∞ we have
$$(F^\infty)^\perp = (F^\perp)^\infty.$$
[Note: When E^∞ is interpreted as ***external tensor product*** $H \otimes E$ (with $\dim H = \#\mathbb{N}$), the formula reads
$$(H \otimes F)^\perp = H \otimes F^\perp.$$
Of course, it holds for arbitrary H, when $H \otimes E$ corresponds to $E^{\dim H}$.]

2. Show:
   (a) $\mathcal{K}(\mathcal{B}, E) = E$.
   (b) $\mathcal{K}(E, F) = \mathcal{K}(F, E)^*$.
   (c) $\mathcal{K}\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \mathcal{K}(E_1) & \mathcal{K}(E_2, E_1) \\ \mathcal{K}(E_1, E_2) & \mathcal{K}(E_2) \end{pmatrix}$.

3. We consider the adjointable operators on the Hilbert $\mathcal{B}$–module E and on the Hilbert $\mathcal{K}(E)$–module $\mathcal{K}(E)$. Show that
$$\mathcal{B}^a(E) \ni a \longmapsto (\alpha(a)\colon xy^* \mapsto (ax)y^*)$$
defines a ($*$–)homomorphism $\alpha\colon \mathcal{B}^a(E) \to \mathcal{B}^a(\mathcal{K}(E))$, and
$$\mathcal{B}^a(\mathcal{K}(E)) \ni a' \longmapsto (\alpha'(a')\colon xy^*z \mapsto (a'xy^*)z)$$
its inverse α'.

4. (a) Verify $(\mathbb{C}^n)^* = \mathbb{C}_n$.
   (b) Show $\mathcal{L}^r(\mathbb{C}_n) = \mathbb{C}\,\mathrm{id}_{\mathbb{C}_n}$.