Problems: “Hilbert modules and their applications”
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No. 4 (discussion November 17)

1. Show that $(E^n)_m$“=”$(E_m)^n$, henceforth denoted by $M_{n,m}(E)$, and that the elements in $M_{n,m}$ are conveniently represented as $n \times m$–matrices $X = (x_{i,j})_{i.j}$ with inner product and right action given by

$$\langle X, Y\rangle_{i,j} = \sum_k \langle x_{k,i}, y_{k,j}\rangle, \qquad (XB)_{i,j} = \sum_k x_{i,k} b_{k,j}.$$

If E is a correspondence from $\mathcal{A}$ to $\mathcal{B}$, then $M_{n,m}(E)$ inherits the structure of a correspondence from $M_n(\mathcal{A})$ to $M_n(\mathcal{B})$ by defining the left action

$$(AX)_{i,j} = \sum_k a_{i,k} x_{k,j}.$$

2. Let E be an $\mathcal{A}$–$\mathcal{B}$–correspondence. Show that $E \odot \mathcal{B} = \mathcal{B}$ and $\mathcal{A} \odot E = E$.

3. In computations in the tensor product of ${}_{\mathcal{A}}E_{\mathcal{B}}$ and ${}_{\mathcal{B}}F_{C}$, we frequently write an element in $z = \sum_i x_i \odot y_i \in E \odot F$ as a simple tensor $X_n \odot Y^n \in E_n \odot F^n$.

   (a) Show that this, actually, establishes an isomorphism of $\mathcal{A}$–C–correspondences.
   Here and henceforth, we will write $E_n \odot F^n = E \odot F$ for such sort of isomorphisms.

   (b) Show $M_{n.k}(E) \odot M_{k,m}(F) = M_{n.m}(E \odot F)$.

   (c) Show $E^n \odot \mathcal{B}_m = M_{n,m}(E) = \mathcal{A}^n \odot E_m$.

   (d) Show $M_{n,m}(E) = M_{n,m} \otimes E$.

4. For $i = 1, 2$, let E_i be a Hilbert $\mathcal{B}_i$–module. Choose any C^*–norm on $\mathcal{B}_1 \underline{\otimes} \mathcal{B}_2$ and denote by $\mathcal{B}_1 \otimes \mathcal{B}_2$ the C^*–algebra obtained by completion. Show:

   (a) The Hilbert $\mathcal{B}_1 \otimes \mathcal{B}_2$–module $\mathcal{B}_1 \otimes \mathcal{B}_2$ may be viewed as a correspondence from $\mathcal{B}_1$ to $\mathcal{B}_1 \otimes \mathcal{B}_2$ via the left action of $\mathcal{B}_1$ determined by $b_1'(b_1 \otimes b_2) = b_1' b_1 \otimes b_2$.

   (b) The tensor product $E_1 \odot (\mathcal{B}_1 \otimes \mathcal{B}_2)$ may be viewed as a correspondence from $\mathcal{B}_2$ to $\mathcal{B}_1 \otimes \mathcal{B}_2$ via the left action of $\mathcal{B}_2$ determined by $b_2'(x_1 \odot (b_1 \otimes b_2)) = x_1 \odot (b_1 \otimes b_2' b_2)$.

   (c) The tensor product $E_2 \odot (E_1 \odot (\mathcal{B}_1 \otimes \mathcal{B}_2))$ is (“canonically”) isomorphic to $E_1 \otimes E_2$.

[Notes: - Observe that the tensor product has no meaning, if we wish to compute first “$E_1 \odot E_2$”. Associativity of the tensor product only holds/makes sense, if the correspondences are correspondences over the right algebras, from the beginning. Here, we define the action of $\mathcal{B}_2$ that enables the second tensorproduct, only *ex post* after having computed the first.

- If each E_i is a correspondence from $\mathcal{A}_i$ to $\mathcal{B}_i$, by general theory, the internal tensor product in (c) is a correspondence from $\mathcal{A}_2$ to $\mathcal{B}_1 \otimes \mathcal{B}_2$. Of course, in the picture $E_1 \otimes E_2$, this action of $\mathcal{A}_2$ coincides with the canonical one. By symmetry, the same is true, exchanging E_1 and E_2. So, $E_1 \otimes E_2$ carries actions of $\mathcal{A}_1$ and $\mathcal{A}_2$ and, moreover, these actions “commute” ($a_1 a_2 z = a_2 a_1 z$) and there is a homomorphism from $\mathcal{A}_1 \underline{\otimes} \mathcal{A}_2$ into $\mathscr{B}^a(E_1 \otimes E_2)$. Wether or not this homomorphism extends to some C^*–completion $\mathcal{A}_1 \otimes \mathcal{A}_2$, depends on both norms. However the answer is always affirmative if on $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the minimal (or *spatial*) C^*–tensor product or if $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the maximal C^*–tensor product. In the former case, the norm on $\mathscr{B}^a(E_1) \underline{\otimes} \mathscr{B}^a(E_2)$ is, in fact, the spatial norm.]