Problems: "Hilbert modules and their applications"
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No. 5 (discussion November 24)

1. Let E be a Hilbert $\mathcal{B}$–module and let F be a correspondence from $\mathcal{B}$ to $\mathcal{C}$. Recall that a correspondence is ***faithful*** if the left action defines a faithful homomorphism. (We also say, 'the left action is faithful'.)

   Show:

   (a) If E is full, then $x \odot y = x \odot y'$ for all $x \in E$ implies $y = y'$.

   (b) If F is faithful, then $x \odot y = x' \odot y$ for all $y \in F$ implies $x = x'$.

2. Find an example of two $\mathcal{B}$–correspondences(=correspondence from $\mathcal{B}$ to $\mathcal{B}$) E and F such that

$$E \odot F = \{0\}, \qquad\qquad F \odot E \neq \{0\}.$$

3. Consider the Hilbert $\mathcal{B}$–module E as correspondence from $\mathcal{K}(E)$ to $\mathcal{B}_E$ and establish the following isomorphisms of correspondences (over $\mathcal{B}_E$ and over $\mathcal{K}(E)$, respectively).

$$E^* \odot E = \mathcal{B}_E, \qquad\qquad E \odot E^* = \mathcal{K}(E).$$

4. Let E be a Hilbert $\mathcal{B}$–module and let F be a correspondence from $\mathcal{B}$ to $\mathcal{C}$. Show:

   (a) $a \mapsto (\mathsf{id}_E \odot a \colon x \odot y \mapsto x \odot ay)$ defines a homomorphism $\mathcal{B}^{a,bil}(F) \to \mathsf{id}_E \odot \mathcal{B}^{a,bil}(F) \subset \mathcal{B}^a(E \odot F)$.

   (b) If E is full (so that $E^* \odot E \odot F = F$), then this homomorphism is an isomorphism onto the the relative of commutant of $\mathcal{B}^a(E) \odot \mathsf{id}_F$ in $\mathcal{B}^a(E \odot F)$ (that is, onto the set of all adjointable operators on $E \odot F$ that commute with all $a' \odot \mathsf{id}_F$).

   [In particular, if $F = \mathcal{B}$ so that $E \odot F = E$, then we get an isomorphism of the center of $\mathcal{B}^a(E)$ onto the center of $\mathcal{B}^a(\mathcal{B}) = M(\mathcal{B})$.]

   [Note: The proof of the statement in Theorem 4.2.18 and Observation 4.2.20 in my Habilitation, that $a \mapsto \mathsf{id}_E \odot a$ is a contraction even if all objects $E, F, \mathcal{B}, \mathcal{C}$ are not assumed complete and even if the left action of $\mathcal{B}$ on F is not assumed contractive, is false. (It only works if $\mathcal{B}$ is a C^*–algebra, or if the left action of $\mathcal{B}$ on F is assumed contractive explicitly.)

   Right now I don't know if the statement itself is false, or not. (I never used it ....)]