Other exercises to: "Hilbert modules and their applications"
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1. A linear map $T: \mathcal{A} \to \mathcal{B}$ is ***completely positive*** (***CP***) if

$$\sum_{i,j} b_i^* T(a_i^* a_j) b_j \;\geq\; 0$$

for all finite choices $a_i \in \mathcal{A}, b_i \in \mathcal{B}$. The map T is ***n–positive*** if the ***inflation*** $T_n\colon M_n(\mathcal{A}) \to M_b(\mathcal{B})$ (that is, T acting matrix element-wise) is positive.

   Show: T is CP if and only if T is n–positive for all $n \in \mathbb{N} := \{1, 2, \ldots\}$. (The latter is the "usual" definition of CP.) Also, in either case T_n is CP for all $n \in \mathbb{N}$.

2. $$\langle x, y \rangle \;=\; \frac{1}{4} \sum_{k=0}^{3} i^k \langle i^k x + y, i^k x + y \rangle$$

3. In the lecture we have shown that a semiinner product is symmetric ($\langle x, y \rangle = \langle y, x \rangle^*$) automatically. Is this true also for real vector spaces? To be precise: Suppose we have a real vector space V with a bilinear map $(\bullet, \bullet)\colon V \times V \to \mathbb{R}$ that is positive ($(x, x) \geq 0$). Is it true that always $(x, y) = (y, x)$?

October 21

4. Let $\mathcal{A}$ and $\mathcal{B}$ be $*$–algebras and $T\colon \mathcal{A} \to \mathcal{B}$ a linear map. Make the statement "the prescription $\langle a \otimes b, a' \otimes b' \rangle := b^* T(a^* a') b'$ defines a sesquilinear map $\langle \bullet, \bullet \rangle$ on $\mathcal{A} \otimes \mathcal{B}$" precise, and prove that statement.

5. Let $\mathcal{A}, \mathcal{B}, C$ be algebras, let E be a right $\mathcal{B}$–module, and let F be a left $\mathcal{B}$–module. Define $\mathcal{N}_{\mathcal{B}} := \mathsf{span}\{xb \otimes y - x \otimes by\colon x \in E, b \in \mathcal{B}, y \in F\}$, the subspace of $E \otimes F$ generated by the "relations" $xb \otimes y - x \otimes by$. Recall that a blinear map $j\colon E \times F \to V$ (into a vector space V) is ***balanced*** if $j(xb, y) = j(x, by)$.

   (a) The pair $(E \odot F := (E \otimes F)/\mathcal{N}_{\mathcal{B}}, i\colon (x, y) \mapsto x \odot y := x \otimes y + \mathcal{N}_{\mathcal{B}})$ consisting of a vector space $E \odot F$ and a balanced bilinear map $i\colon E \times F \to E \odot F$ enjoys the following ***universal property***: For every bilinear balanced map $j\colon E \times F \to V$, there is a unique linear map $\widehat{j}\colon E \odot F \to V$ such that $\widehat{j}(x \odot y) = j(x, y)$.

   (b) If $(E \odot' F, i')$ is another pair enjoying that universal property, then $x \odot y \mapsto x \odot' y$ determines a unique isomorphism $u\colon E \odot F \to E \odot' F$ such that $ui = i'$. (That's why the property is called *universal*.)

   (c) If is E an $\mathcal{A}$–$\mathcal{B}$–bimodule and if F is a $\mathcal{B}$–C–bimodule, then $E \odot F$ "inherits" an $\mathcal{A}$–C–bimodule structure via $a(x \odot y)c := (ax) \odot (yc)$. Prove that. Formulate and prove the analogue universal property for balanced $\mathcal{A}$–C–bilinear mas j (that is, apart from being bilinear and balanced, j also fulfills $j(ax, yc) = aj(x, y)c$).

October 22

6. (a) The pre-Hilbert module direct sum of a finite number of Hilbert modules is always complete.

   (b) The pre-Hilbert module direct sum of infinitely many nonzero pre-Hilbert modules is never complete.

   (c) The Hilbert module direct sum $\bigoplus_{i\in\mathcal{I}} E_i$ of Hilbert modules E_i consists of all families $(x_i)_{i\in\mathcal{I}}$ $(x_i \in E_i)$ such that the sum $\sum_{i\in\mathcal{I}}\langle x_i, x_i\rangle$ converges in $\mathcal{B}$ over the finite subsets of $\mathcal{I}$.

   (Observe that projections onto "direct subsums" behave nicely.)

7. If L is a closed left ideal in a C^*–algebra $\mathcal{A}$, then $\overline{\mathsf{span}}\, L^*L = L \cap L^*$. (For the non-obvious direction, see the supplement in Problem 1.1.)

8. If a closed subspace E of a C^*–algebra $\mathcal{A}$ is a ***ternary subspace*** (that is, if $xy^*z \in E$ for all $x, y, z \in E$), then E is Hilbert module over the C^*–algebra $\mathcal{B} := \overline{\mathsf{span}}\, E^*E$.

9. Suppose S is a subspace of the right $\mathcal{B}$–module E that generates E as a right $\mathcal{B}$–module, that is, $\mathsf{span}\, S\mathcal{B} = E$. Suppose $a\colon S \to F$ is a map into another right $\mathcal{B}$–module F. Then a extends to a (unique!) $\mathcal{B}$–linear map $E \to F$ if and only if $\sum_{\sigma\in S} \sigma b_\sigma = 0 \Rightarrow \sum_{\sigma\in S} a(\sigma)b_\sigma = 0$ (only finitely many $b_\sigma \neq 0$).

October 27

10. Find a pre-Hilbert space H such that no two of the spaces $\mathcal{L}(H)$, $\mathcal{B}(H)$, $\mathcal{L}^a(H)$, $\mathcal{B}^a(H)$ coincide.

11. If H is a Hilbert space, then $H^* = \mathcal{B}(H, \mathbb{C})$ is a Hilbert module over the compact operators on H, $\mathcal{K}(H)$. (It's a ternary subspace of $\mathcal{B}\begin{pmatrix}\mathbb{C}\\ H\end{pmatrix}$.) This $\mathcal{K}(H)$–module is *self-dual*, that is, $\mathcal{B}^r(H^*, \mathcal{K}(H)) = (H^*)^* = H = \mathcal{B}(\mathbb{C}, H)$.

12. If $\mathcal{I}$ is an ideal in $\mathcal{B} \ni \mathbf{1}$ and if $\Phi \in \mathcal{B}^r(\mathcal{I}, \mathcal{B})$ is not adjointable, then $\begin{pmatrix}0 & \Phi\\ 0 & 0\end{pmatrix} \in \mathcal{B}^r\begin{pmatrix}\mathcal{B}\\ \mathcal{I}\end{pmatrix}$ is not adjointable, too.

13. (See (6c).) Let $\mathfrak{n}$ be a cardinal number and choose a set $\mathcal{I}$ such that $\#\mathcal{I} = \mathfrak{n}$. We define $\mathcal{B}^{\mathfrak{n}} := \bigoplus_{i\in\mathcal{I}} \mathcal{B}$. (It is worth a moments thought, to ask in which sense this does not depend on the choice of $\mathcal{I}$.) If $\mathcal{B}$ is unital, then the bounded right $\mathcal{B}$–linear maps from $\mathcal{B}^{\mathfrak{n}}$ to $\mathcal{B}$ are

$$\mathcal{B}^r(\mathcal{B}^{\mathfrak{n}}, \mathcal{B}) = \Big\{(b_i)_{i\in\mathcal{I}} \,\Big|\, b_i \in \mathcal{B}, \exists\, M\colon \Big\|\textstyle\sum_{i\in\mathcal{I}'} b_i^* b_i\Big\| \le M\, \forall\, \mathcal{I}' \subset \mathcal{I}, \#\mathcal{I}' < \infty\Big\}.$$

Find $\mathcal{B}$ and $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{B}^r(\mathcal{B}^\infty, \mathcal{B})$ but not in $(\mathcal{B}^\infty)^*$.

[Supplement: In the situation of (6c),

$$\mathcal{B}^r\Big(\textstyle\bigoplus_{i\in\mathcal{I}} E_i, \mathcal{B}\Big) \supset \Big\{(x_i)_{i\in\mathcal{I}} \,\Big|\, x_i \in E_i, \exists\, M\colon \Big\|\textstyle\sum_{i\in\mathcal{I}'}\langle x_i, x_i\rangle\Big\| \le M\, \forall\, \mathcal{I}' \subset \mathcal{I}, \#\mathcal{I}' < \infty\Big\}.$$

Can you find a (necessary and) sufficient condition for equality?]

14. ....