Let H be a Hilbert space. On $\mathcal{B}(H)$, there is a whole zoo of topologies weaker than the norm topology – and all of them are considered when it comes to von Neumann algebras. It is, however, a good idea to concentrate on one of them right from the definition. My choice – and Murphy's [Mur90, Chapter 4] – is the strong (or strong operator=STOP) topology:

Definition. A ***von Neumann algebra*** is a $*$–subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ of operators acting nondegenerately(**!**) on a Hilbert space H that is strongly closed in $\mathcal{B}(H)$.

(Every norm convergent sequence converges strongly, so $\mathcal{A}$ is a C^*–algebra.)

This does not mean that one has not to know the other topologies; on the contrary, one has to know them very well, too. But it does mean that proof techniques are focused on the strong topology; if we use a different topology to prove something, then we do this only if there is a specific reason for doing so.

One reason why it is not sufficient to worry only about the strong topology, is that the strong topology (unlike the norm topology of a C^*–algebra) is not determined by the algebraic structure alone: There are "good" algebraic isomorphisms between von Neumann algebras that do not respect their strong topologies. A striking feature of the strong topology on $\mathcal{B}(H)$ is that $\mathcal{B}(H)$ is order complete:

Theorem (Vigier). *If $(a_\lambda)_{\lambda\in\Lambda}$ is an increasing self-adjoint net in $\mathcal{B}(H)$ and bounded above ($\exists c \in \mathcal{B}(H)\colon a_\lambda \le c \forall\lambda$), then a_λ converges strongly in $\mathcal{B}(H)$, obviously to its least upper bound in $\mathcal{B}(H)$.*

(One may simply restrict to bounded nets, by passing to the subnet indexed by $\{\lambda \ge \lambda_0\}$ for some λ_0 so that now $a_{\lambda_0} \le a_\lambda \le c$.)

Since the strong limit of a net in a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ is again in $\mathcal{A}$, the order completeness turns over to $\mathcal{A}$.

Corollary. *A von Neumann algebra is unital. In fact, the least upper bound of the standard approximate unit (of any C^*–subalgebra of $\mathcal{B}(H)$ that acts nondegenerately) has no choice but being id_H.*

The order structure of any C^*–algebra **is** determined by its algebraic structure alone. (In fact, a is positive if and only if it can be written as b^*b.)

Definition. A positive(**!**) linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ between von Neumann algebras is ***normal*** if it is ***order continuous***, that is, if $l.u.b.\, \varphi(a_\lambda) = \varphi(l.u.b.\, a_\lambda)$ for each increasing bounded net.

We see: Algebraic isomorphisms between von Neumann algebras are normal.

Also, normality is a matter of bounded subsets.

We shall see later on how normality relates to continuity in certain topologies.

Elements $a \in \mathcal{B}(H)$ allow ***polar decomposition*** $a = v\,|a|$ where v is the uniquely determined by this equation and the condition $\ker v = \ker a$, and bound to be a partial isometry. It is important to note that if a is from a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$, then, apart from $|a|$, also v is in $\mathcal{A}$. (If $a = a_\alpha\,|a|^\alpha$ with unique $a_\alpha \in \overline{aC^*(|a|)}$, then $a_\alpha \xrightarrow{\alpha\uparrow 1} v$, strongly.)

Let S be a subset of $\mathcal{B}(H)$. By $\overline{S}^s$ we denote its strong closure. Define the commutant ***commutant*** os S as $S' := \{a \in \mathcal{B}(H)\colon as = sa\ (s \in S)\}$. Clearly, S' is a strongly closed , unital (and, therefore, nondegenerate) subalgebra of $\mathcal{B}(H)$. If $S = S^*$ (or, weaker, if $(\overline{\operatorname{span}}^s S)^* = \overline{\operatorname{span}}^s S$), then S' is a von Neumann algebra on H. Exercise: Like for orthogonal complements in Hilbert spaces, we have $S'' \supset S$ and $S''' = S'$.

For a pre-Hilbert subspace D of a Hilbert space H we have $D^{\perp\perp} = \overline{\operatorname{span}}\,D$. Is the same true for $*$–subalgebras of $\mathcal{B}(H)$? **The** basic theorem on von Neumann algebras is von Neumann's ***double commutant theorem***. It has two versions – and the second is by far more important, because it answers our question in the affirmative sense.

Theorem. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a $*$–algebra of operators acting on the Hilbert space H. Then:*

1. *$\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*
2. *If $\mathcal{A}$ acts non degenerately, then $\mathcal{A}'' = \overline{\mathcal{A}}^s$.*

Note: In particular, $\overline{\mathcal{A}}^s$ is an algebra! Since operator multiplication is not jointly strongly continuous, it is far from evident why the product of two elements in the strong closure is again in the strong closure. One way out would be the ***Kaplansky density theorem*** ....

Theorem. *If $\mathcal{A}$ is a $*$–subalgebra of $\mathcal{B}(H)$, then the ball of $\mathcal{A}$ is strongly dense in the ball of $\overline{\mathcal{A}}^s$.*

(Likewise for the self-adjoint elements and for the balls in the self-adjoint elements and, if $\mathcal{A}$ is unital, for the unitaries. Actually, the statements about self-adjoint elements is proved first.)

.... if this theorem was not usually proved after the double commutant theorem. It is important for other reasons, for instance, if we wish to control norms in sets of products (like simple tensors in tensor products).

Exercise: Examine what happens in the double commutant theorem if we drop in (2) non-degeneracy.

A strongly closed ideal $\mathcal{I}$ in a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ is a von Neumann algebra (though acting on the Hilbert space $\overline{\operatorname{span}}\,\mathcal{I}H$, only). Therefore, it is unital, with unit q, say. Since for every $a \in \mathcal{A}$, we have $\mathcal{I} \ni qa = qaq = aq \in \mathcal{I}$, we see that q is a ***central*** projection. In fact, the central projections and the strongly closed ideals are in one-to-one correspondence via $q \leftrightarrow q\mathcal{A} = \mathcal{A}q$. An analogue argument for strongly closed hereditary subalgebras of $\mathcal{A}$ shows that strongly closed right (left) ideals are in one-to-one correspondence with projections p (not necessarily central) via $p \leftrightarrow p\mathcal{A}$ $\;(p \leftrightarrow \mathcal{A}p)$.

We meticulously avoided the question if with a subset S also the subset S^* is strongly closed. In general, the answer is no; however, for convex subsets it is yes. A way to seeing this, is an instance of where we can no longer avoid other topologies. Every topology on $\mathcal{B}(H)$ gives rise to the relative topology on the subset $\mathscr{A}$, and, as topology on $\mathscr{A}$, will carry the same name.
The weak operator (=WOP) topology is generated by seminorms $|\langle x, \bullet y\rangle|$. Obviously, it is weaker than the strong topology, and, unlike the strong topology, it is $*$–invariant. We collect some properties, relating it also to the strong topology.

Proposition.

1. *A convex subset of $\mathcal{B}(H)$ (for instance, a subspace) is strongly closed if and only if it is weakly closed. Therefore, a weakly closed subset is $*$–strongly closed.*

2. *The weakly continuous linear functionals are precisely the elements of* $\mathsf{span}\{\langle x, \bullet y\rangle\}$.

3. *The strongly continuous and weakly continuous linear functionals are the same. Therefore, since the continuous functionals in a duality determine the topology, and since strong and weak topology do not coincide (for instance, because the strong topology is not $*$–invariant), the strong topology cannot be induced via a duality.*

The ***trace*** on $\mathcal{B}(H)$ is defined as $\mathsf{tr}\, a := \sum_i \langle e_i, ae_i\rangle$ "whenever it reasonably exists", where (e_i) is any ONB of H (it does not depend on the choice). The trace is **the** example of a *semi-finite, normal weight* on a von Neumann algebra, a concept that carries that positive functionals to possibly unbounded (and, therefore, not everywhere defined) positive functionals. Weights are defined only on a (sufficiently dense subset of) the positive elements. An operator $a \in \mathcal{B}(H)$ is ***trace class*** if $\mathsf{tr}\,|a| < \infty$. The set of trace class operators on H is denoted by $L^1(H)$.

Once in a while, by * attached to a space we shall mean its topological dual. In the lecture they won't occur. Only the pre-dual $_*$ survives.

Proposition [Mur90, Sections 2.4 and 4.2].

1. *The **trace norm** $\|a\|_1 := \mathsf{tr}\,|a|$ is a $*$–invariant (improper) norm on $\mathcal{B}(H)$. Therefore, $L_1(H)$ with $\|\bullet\|_1$ is an involutive normed space.*

2. *We have $\mathcal{F}(H) \subset L^1(H) \subset \mathcal{K}(E)$ and $\mathcal{F}(H)$ is dense in $L^1(H)$ for $\|\bullet\|_1$.*

3. *The trace* $\mathsf{tr}\, a$ *exists absolutely for all $a \in L^1(H)$.*

4. *$\|ab\|_1 \leq \|a\|_1 \|b\|$ so that $L^1(H)$ is an ideal in $\mathcal{B}(H)$.*

5. *$\mathcal{K}(H)^* = L^1(H)$ and $L^1(H)^* = \mathcal{B}(H)$ under the duality $(a, b) \mapsto \mathsf{tr}(ab)$ for $a \in L^1(H)$ and $b \in \mathcal{K}(H)$ or $a \in \mathcal{B}(H)$. In particular, $L^1(H)$ is complete.*

The weak* topology induced on $\mathcal{B}(H)$ by being the dual space of $L^1(H)$ is called the ***σ–weak*** (or ***ultra weak***) ***topology***. Since $\mathcal{F}(H) \subset L^1(H)$ (this includes that $\mathrm{tr}((xy^*)a) = \langle y, ax\rangle$), the σ–weak topology is stronger than the weak topology. It is not comparable to the ($*$–)strong topology. The σ–weakly continuous or just ***σ–weak*** linear functionals are precisely the elements of $L^1(H)$. Therefore:

Proposition. *For checking σ–weak convergence on a bounded subset, it is sufficient to check convergence with functionals*

1. *$\langle x, \bullet y\rangle$ with x, y from a total subset of H, or*

2. *$\langle x, \bullet x\rangle$ with x from a dense subset of H.*

Observation. By definition, a is trace class if $|a|$ is trace class. It follows that with $a = a^*$ also a_+ and a_- are trace class. Moreover, the functional $\mathrm{tr}(a\bullet)$ is positive, if (and only if, as long as we are speaking about $\mathcal{B}(H)$) a is positive. Therefore, each $L^1(H)$ is a linear combination of four positive σ–weak functionals.

Lemma [Mey93, Appendix 4.3.5+6]. *Let $\mathcal{A} \subset \mathcal{B}(H)$ a von Neumann algebra.*

1. *A positive σ–weak functional on $\mathcal{A}$ extends to a positive σ–weak functional on $\mathcal{B}(H)$. (Uses diagonalization of self-adjoint elements of $\mathcal{K}(H)$.)*

2. *A positive functional on $\mathcal{A}$ is σ–weak if and only if it is normal. (Uses Zorn.)*

Corollary. *Being σ–weak is a matter of bounded subsets and, therefore, may be checked as in the preceding proposition.*

Theorem.

1. *Every von Neumann algebra is the dual $\mathcal{A} = (\mathcal{A}_*)^*$ of its* ***pre-dual*** *$\mathcal{A}_* := L^1(H)/\mathcal{N}$, where $\mathcal{N} := \ker(\varphi \mapsto \varphi \upharpoonright \mathcal{A})$.*

2. *The pre-dual is unique as a Banach space. ($\mathcal{A} = B^* \Rightarrow B \overset{isom}{\cong} \mathcal{A}_*$.)*

3. *$\mathcal{A}_*$ consists precisely of the σ–weak linear functionals on $\mathcal{A}$. It is spanned by its normal elements and, therefore, pre-dual and σ–weak topology are intrinsic to the von Neumann algebra $\mathcal{A}$.*

If $\mathbb{K}$ is a separable infinite-dimensional Hilbert space, then the weak and the σ–weak linear functionals on the von Neumann algebra(!) $\mathcal{A} \otimes \mathrm{id}_{\mathbb{K}}$ on $H \otimes \mathbb{K}$ coincide, and since the pre-dual does not depend on the representation, both coincide with $\mathcal{A}_*$. (Effectively, since every $\varphi \in \mathcal{A}_*$ can be written as $\sum_{n=1}^{\infty}\langle x_i, \bullet y_n\rangle$ for sequences in H with $\sum_n \|x_n\|^2 < \infty$ and $\sum_n \|y_n\|^2 < \infty$, we get $\varphi = \langle x, (\bullet \otimes \mathrm{id}_{\mathbb{K}})y\rangle$ for suitable $x, y \in H \otimes \mathbb{K}$.) The strong topology in this representation is called the ***σ–strong topology***; it is intrinsic, too.

Just for completeness (not that we needed it ....): A W^****–algebra*** is a C^*–algebra that admits a pre-dual. So, every von Neumann algebra is a W^*–algebra. Conversely, every W^*–algebra is (algebraically) isomorphic to a von Neumann algebra (via a weak* continuous, hence normal, monomorphism into some $\mathcal{B}(H)$). See Sakai [Sak71].

Let us close with some results:

[Ske16, Lemma B.2] (Dini's theorem for nets). *For each $t \in [a, b]$ let φ_t a normal (hence, positive) linear functional on the von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$, such that for each $a \in \mathcal{A}$ the function $t \mapsto \varphi_t(a)$ is continuous. Let (a_λ) be a net in $\mathcal{A}$ increasing to $a \in \mathcal{A}$. Then for every $\varepsilon > 0$ there exists λ_0 such that $\varphi_t(a) - \varphi_t(a_\lambda) < \varepsilon$ for all t and all $\lambda \geq \lambda_0$.*

[Ske16, Lemma A.2] (joint weak continuity for semigroups). *Let $T = (T_t)_{t \in \mathbb{R}_+}$ by a weakly continuous one-parameter semigroup of weakly continuous maps T_t on a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$. Then for each bounded subset B of $\mathcal{A}$, the map $(t, a) \mapsto T_t(a)$ from $\mathbb{R}_t \times B$ to $\mathcal{A}$ is jointly continuous for the (relative) (σ–)weak topologies either side.*

The latter result passes to the ***pre-dual semigroup*** $(T_t)_*\colon \mathcal{A}_* \to \mathcal{A}_*$ defined by $\varphi \mapsto (a \mapsto \varphi \circ T_t(a))$, and, then, applying to that strongly continuous semigroup standard result about C_0–semigroups. A really excellent reference for such semigroup is the little book by Engel and Nagel [EN06]. (Well, at least as long continuity is concerned. Who wishes to study measurable semigroups, has to use the classics [HP57] by Hille and Phillips.)

References

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Appendix: [BDH88]'s σ– and s–topologies. By passing to $\mathcal{A} \otimes \mathsf{id}_{\mathbb{K}} \subset \mathcal{B}(H \otimes \mathbb{K})$ if necessary, we may assume that $\mathcal{A} \subset \mathcal{B}(H)$ is ***spatially stable***, that is, that $\mathcal{A}$ and $\mathcal{A} \otimes \mathsf{id}_{\mathbb{K}}$ are unitarily equivalent. Recall that the weak and σ–weak functionals on such $\mathcal{A}$ coincide; see bottom of p.4.

Definition [BDH88]. Let E be a pre-Hilbert module over a W^*–algebra $\mathcal{B}$.

1. The ***σ–topology*** of E is generated by seminorms $\sum_{n\in\mathbb{N}} \varphi_n \circ \langle x_n, \bullet \rangle$ with $\varphi_n \in \mathcal{B}_*$ and $x_n \in E$ such that $\sum_{n\in\mathbb{N}} \|\varphi_n\| \, \|x_n\| < \infty$.
2. The ***s–topology*** of E is generated by seminorms $\sqrt{\varphi(|\bullet|^2)}$ with $\varphi \in \mathcal{B}_*$.

Proposition. *Suppose $\mathcal{B} \subset \mathcal{B}(G)$ is a von Neumann algebra.*

1. *The σ–topology coincides with the (relative) σ–weak topology of $\overline{E}^s (\supset E)$.*
2. *The s–topology coincides with the (relative) σ–strong topology of $\overline{E}^s (\supset E)$.*

Proof. Assume $\mathcal{B}$ is spatially stable, and recall that σ–weak and σ–strong topology do not depend on how we represented $\mathcal{B}$.

(1) By shifting the factor $\|x_n\|$ over to φ_n (provided $\varphi_n \neq 0$; otherwise cancel that summand), we may assume that $\|x_n\| = 1$. Every φ_n has the form $\sum_i \langle \omega_{n,i} \bullet \omega'_{n,i} \rangle$ for a vectors $\omega_{n,i}, \omega'_{n,i}$ in G with $\sum_i \|\omega_{n,i}\| \, \|\omega'_{n,i}\| = \|\varphi_n\|$. We may assume that $\|\omega_{n,i}\| = \|\omega_{n,i}\| = \sqrt{\|\omega_{n,i}\| \, \|\omega'_{n,i}\|} =: c_{n,i}$. Note that $c_{n,i}$ is square-summable over (n, i). Stabilizing once more, we find that the vector $h \in H := E \odot G$ that corresponds to $\sum_{n,i} (x_n \odot (\omega_{n,i} \otimes e_{n,i})$ in $E \odot (G \otimes \mathbb{K}) \cong E \odot G = H$ and the vector $g \in G$ that corresponds to $\sum_{n,i} \omega_{n,i} \otimes e_{n,i}$ in $G \otimes \mathbb{K} \cong G$ ($(e_{n,i})$ some ONB of $\mathbb{K}$), fulfill $\sum_{n\in\mathbb{N}} \varphi_n \circ \langle x_n, \bullet \rangle = \langle h, \bullet g \rangle$. Therefore the σ–topology is weaker than the σ–weak topology.

On the other hand, the functionals $\varphi \langle x, \bullet \rangle$ form a total subset of the weak, hence, of the σ–weak functionals. [BDH88] argue that the space of functionals they use is complete. Therefore, contains all σ–weak functionals, so that the σ–topology is also stronger than the σ–weak topology.

(2) The proof for s–topology and σ–strong topology is very similar after recalling that also the σ–strong topology of $\mathcal{B}$ is generated by seminorms $\sqrt{\varphi(|\bullet|)^2}$. ■

References

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