UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 1
for the tutorial on *Tuesday, 4 November*

Exercise 1. Let $(\mathcal{A}, \varphi)$ be a C^*-probability space and let $X \in \mathcal{A}$.

(a) Let X be self-adjoint ($X = X^*$). Show that there is a uniquely determined compactly supported probability measure μ on $\mathbb{R}$ such that:

$$\int_{\mathbb{R}} t^k d\mu(t) = \varphi(X^k) \qquad \forall k \in \mathbb{N}$$

In this sense, we identify the *(non-comm.) distribution of* X and the measure μ.

(b) Let X be normal ($XX^* = X^*X$). What are the moments of X with respect to φ? Show that again, we may identify the distribution of X with a uniquely determined compactly supported probability measure μ on $\mathbb{C}$.

(c) Let $(\Omega, \Sigma, \mathbb{P})$ be a (classical) probability space. Show that the space $\mathcal{A} := L^\infty(\Omega, \mathbb{P})$ of all real-valued random variables together with $\varphi(X) := \mathbb{E}(X) := \int X(\omega) d\mathbb{P}(\omega)$ is a non-commutative probability space in the sense of Definition 1.1 of the lecture. In $\mathcal{A}$ all random variables commute. In the classical theory, the measure μ from (a) is called the distribution of a random variable $X \in \mathcal{A}$. This justifies the name *non-commutative distribution* used in the lecture.

Exercise 2. Let (A, φ) be a C^*-probability space and let $u \in A$. We say that u is a *Haar unitary*, if it is a unitary ($u^*u = uu^* = 1$) and $\varphi(u^k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Here, we use the convention $u^{-k} = (u^*)^k$ for all $k \in \mathbb{N}$. Furthermore, consider the *free group* $\mathbb{F}_n$ on n generators $a_1, \ldots, a_n$ and denote by $\mathbb{C}\mathbb{F}_n$ its *group (*-)algebra*, i.e.:

$$\mathbb{C}\mathbb{F}_n := \{\sum_{g \in \mathbb{F}_n} \alpha_g g \mid \alpha_g \in \mathbb{C}, \text{only finitely many } \alpha_g \neq 0\} \text{ with formal addition and}$$

$$\Big(\sum \alpha_g g\Big) \cdot \Big(\sum \beta_h h\Big) := \sum_{g,h} \alpha_g \beta_h gh, \qquad \Big(\sum \alpha_g g\Big)^* := \sum \bar{\alpha}_g g^{-1}$$

(a) Show that $(\mathbb{C}\mathbb{F}_n, \tau)$ is a *(non-comm.) *-probability space* where $\tau\left(\sum \alpha_g g\right) := \alpha_e$ (here e is the neutral element), i.e. we also have $\tau(x^*x) \geq 0$. You may use that $\mathbb{C}\mathbb{F}_n$ is a *-*algebra*. Show that any a_i, $i = 1, \ldots, n$ is a Haar unitary.

(b) Let u be a Haar unitary with spectrum $\sigma(u) = S^1 := \big\{z \in \mathbb{C} \;\big|\; |z| = 1\big\}$. By Exercise 1, its distribution is given by a measure μ. Find it. Moreover, show that the identity function $z \mapsto z$ in $\mathcal{C}(S^1)$ is a Haar unitary with respect to it (i.e. we have a model of a Haar unitary in $\mathcal{C}(S^1)$).

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Exercise 3. Let H be a Hilbert space and let $\mathcal{F}(H)$ be its full Fock space as defined in Example 1.7 of the lecture.

(a) Let $f \in H$. Show that the creation operator $l(f)$ is bounded and compute its norm. Show that the annihilation operator $l^*(f)$ is the adjoint of $l(f)$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{F}(H)$.

(b) Let H be one-dimensional and let $e \in H$ be a unit vector. Show that $\mathcal{F}(H)$ is naturally isomorphic to the Hilbert space $\ell^2(\mathbb{N}_0)$ and that $l(e)$ is isomorphic to the unilateral shift $S \in B(\ell^2(\mathbb{N}_0))$ given by $Se_n := e_{n+1}$.

(c) Let H be n dimensional with orthonormal basis $e_1, \ldots, e_n$. Show that the elements $l(e_i)$ fulfill the *Cuntz relations*, i.e.:

$$l^*(e_i)l(e_j) = \delta_{ij}, \qquad 1 - \sum l(e_i)l^*(e_i) = \text{projection onto } \mathbb{C}\Omega$$

Furthermore, show that the operators $s_i := l(e_i) + l^*(e_i)$, $i = 1, \ldots, n$ do not commute.