Assignments for the lecture on Non-Commutative Distributions
Winter term 2014/2015

Assignment 1
for the tutorial on Tuesday, 4 November

Exercise 1. Let \((A, \varphi)\) be a \(C^*\)-probability space and let \(X \in A\).

(a) Let \(X\) be self-adjoint \((X = X^*)\). Show that there is a uniquely determined compactly supported probability measure \(\mu\) on \(\mathbb{R}\) such that:
\[
\int_{\mathbb{R}} t^k d\mu(t) = \varphi(X^k) \quad \forall k \in \mathbb{N}
\]
In this sense, we identify the (non-comm.) distribution of \(X\) and the measure \(\mu\).

(b) Let \(X\) be normal \((XX^* = X^*X)\). What are the moments of \(X\) with respect to \(\varphi\)? Show that again, we may identify the distribution of \(X\) with a uniquely determined compactly supported probability measure \(\mu\) on \(\mathbb{C}\).

(c) Let \((\Omega, \Sigma, \mathbb{P})\) be a (classical) probability space. Show that the space \(A := L^\infty(\Omega, \mathbb{P})\) of all real-valued random variables together with \(\varphi(X) := \mathbb{E}(X) := \int X(\omega) d\mathbb{P}(\omega)\) is a non-commutative probability space in the sense of Definition 1.1 of the lecture. In \(A\) all random variables commute. In the classical theory, the measure \(\mu\) from (a) is called the distribution of a random variable \(X \in A\). This justifies the name non-commutative distribution used in the lecture.

Exercise 2. Let \((A, \varphi)\) be a \(C^*\)-probability space and let \(u \in A\). We say that \(u\) is a Haar unitary, if it is a unitary \((u^*u = uu^* = 1)\) and \(\varphi(u^k) = 0\) for all \(k \in \mathbb{Z}\setminus\{0\}\). Here, we use the convention \(u^{-k} = (u^*)^k\) for all \(k \in \mathbb{N}\).

(a) Show that \((\mathbb{C}^F_n, \tau)\) is a (non-comm.) \(*\)-probability space where \(\tau(\sum \alpha_g g) := \alpha_e\) (here \(e\) is the neutral element), i.e. we also have \(\tau(x^*x) \geq 0\). You may use that \(\mathbb{C}^F_n\) is a \(*\)-algebra. Show that any \(a_i, i = 1, \ldots, n\) is a Haar unitary.

(b) Let \(u\) be a Haar unitary with spectrum \(\sigma(u) = S^1 := \{z \in \mathbb{C} \mid |z| = 1\}\). By Exercise 1, its distribution is given by a measure \(\mu\). Find it. Moreover, show that the identity function \(z \mapsto z\) in \(\mathcal{C}(S^1)\) is a Haar unitary with respect to it (i.e. we have a model of a Haar unitary in \(\mathcal{C}(S^1)\)).

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Exercise 3. Let $H$ be a Hilbert space and let $\mathcal{F}(H)$ be its full Fock space as defined in Example 1.7 of the lecture.

(a) Let $f \in H$. Show that the creation operator $l(f)$ is bounded and compute its norm. Show that the annihilation operator $l^*(f)$ is the adjoint of $l(f)$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{F}(H)$.

(b) Let $H$ be one-dimensional and let $e \in H$ be a unit vector. Show that $\mathcal{F}(H)$ is naturally isomorphic to the Hilbert space $\ell^2(\mathbb{N}_0)$ and that $l(e)$ is isomorphic to the unilateral shift $S \in B(\ell^2(\mathbb{N}_0))$ given by $Se_n := e_{n+1}$.

(c) Let $H$ be $n$ dimensional with orthonormal basis $e_1, \ldots, e_n$. Show that the elements $l(e_i)$ fulfill the Cuntz relations, i.e.:

$$
l^*(e_i)l(e_j) = \delta_{ij}, \quad 1 - \sum_l l(e_i)l^*(e_i) = \text{projection onto } \mathbb{C}\Omega
$$

Furthermore, show that the operators $s_i := l(e_i) + l^*(e_i)$, $i = 1, \ldots, n$ do not commute.