Exercise 1. Consider the algebra $\mathbb{C} \langle x_1, \ldots, x_n \rangle$ of non-commutative polynomials in $n$ non-commuting (formal) variables $x_1, \ldots, x_n$. For $j = 1, \ldots, n$, we denote by

$$\partial_j : \mathbb{C} \langle x_1, \ldots, x_n \rangle \to \mathbb{C} \langle x_1, \ldots, x_n \rangle \otimes \mathbb{C} \langle x_1, \ldots, x_n \rangle$$

the non-commutative derivative with respect to $x_j$. Prove Remark 3.3(2), i.e. prove that

$$p \otimes 1 - 1 \otimes p = \sum_{j=1}^n (\partial_j p \cdot x_j \otimes 1 - 1 \otimes x_j \cdot \partial_j p)$$

holds for any $p \in \mathbb{C} \langle x_1, \ldots, x_n \rangle$.

Exercise 2. Let $\mathcal{B}$ be a unital algebra and consider the algebra $\mathcal{B} \langle x_1, \ldots, x_n \rangle$ of non-commutative polynomials over $\mathcal{B}$ in $n$ non-commuting (formal) variables $x_1, \ldots, x_n$. For $j = 1, \ldots, n$, we denote by

$$\partial_j : \mathcal{B} \langle x_1, \ldots, x_n \rangle \to \mathcal{B} \langle x_1, \ldots, x_n \rangle \otimes \mathcal{B} \langle x_1, \ldots, x_n \rangle$$

the non-commutative derivative over $\mathcal{B}$ with respect to $x_j$.

(a) Let $\mathcal{M}$ be an arbitrary $\mathcal{B} \langle x_1, \ldots, x_n \rangle$-bimodule and let

$$\delta : \mathcal{B} \langle x_1, \ldots, x_n \rangle \to \mathcal{M}$$

be a non-commutative derivation over $\mathcal{B}$ in the sense that we have $\delta(b) = 0$ for all $b \in \mathcal{B}$ and

$$\delta(p_1 p_2) = p_1 \cdot \delta(p_2) + \delta(p_1) \cdot p_1$$

for all $p_1, p_2 \in \mathcal{B} \langle x_1, \ldots, x_n \rangle$,

where $\cdot$ denotes the left and right action, respectively, of $\mathcal{B} \langle x_1, \ldots, x_n \rangle$ on $\mathcal{M}$. Show that for any $p \in \mathcal{B} \langle x_1, \ldots, x_n \rangle$

$$\delta(p) = \sum_{j=1}^n (\partial_j p) \sharp \delta(x_j),$$

where $\sharp : (\mathcal{B} \langle x_1, \ldots, x_n \rangle \otimes \mathcal{B} \langle x_1, \ldots, x_n \rangle) \times \mathcal{M} \to \mathcal{M}$ is defined by bilinear extension of $(p_1 \otimes p_2) \sharp m := p_1 \cdot m \cdot p_2$. 
(b) Let \( R_B \subseteq B\langle x_1, \ldots, x_n \rangle \otimes_B B\langle x_1, \ldots, x_n \rangle \) be the linear subspace spanned by
\[
\{ p_1 b \otimes p_2 - p_1 \otimes b p_2 \mid p_1, p_2 \in B\langle x_1, \ldots, x_n \rangle, b \in B \}.
\]
We consider the quotient space
\[
\mathcal{M} := \frac{B\langle x_1, \ldots, x_n \rangle \otimes_B B\langle x_1, \ldots, x_n \rangle}{R_B},
\]
where we put \( p_1 \otimes_B p_2 := p_1 \otimes p_2 + R_B \). Justify that \( \mathcal{M} \) is a \( B\langle x_1, \ldots, x_n \rangle \)-bimodule and show that \( \delta : B\langle x_1, \ldots, x_n \rangle \to \mathcal{M} \) defined by
\[
\delta(p) := p \otimes_B 1 - 1 \otimes_B p \quad \text{for any } p \in B\langle x_1, \ldots, x_n \rangle
\]
is a non-commutative derivation over \( B \).

(c) Apply the formula obtained in (a) to the derivation considered in (b). Convince yourself that your result reduces in the case \( B = \mathbb{C} \) to the formula that was proven in Problem 1.

(d) Consider \( p \in B\langle x_1, \ldots, x_n \rangle \). Assume that \( \partial_i p = 0 \) for all \( i = 1, \ldots, n \). Show that \( p \in \mathcal{B} \).

**Exercise 3.** Let \( S_1, \ldots, S_n \) be the \( n \) free semicircular elements from Example 1.7 (compare also Theorem 3.14). Fix a natural \( m \). Let \( f : \{1, \ldots, n\}^m \to \mathbb{C} \) be any function that “vanishes on the diagonals”, i.e., \( f(i_1, \ldots, i_m) = 0 \) whenever there are \( k \neq l \) such that \( i_k = i_l \). Put
\[
P := \sum_{i_1, \ldots, i_m = 1}^n f(i_1, \ldots, i_m) S_{i_1} \cdots S_{i_m} \in \mathbb{C}\langle S_1, \ldots, S_n \rangle.
\]
Calculate
\[
\sum_{i=1}^n \partial_i^* \partial_i P.
\]