UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 2B
for the tutorial on *Tuesday, 18 November* (in SR6)

Exercise 1. Consider the algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ of non-commutative polynomials in n non-commuting (formal) variables $x_1, \dots, x_n$. For $j = 1, \dots, n$, we denote by

$$\partial_j : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

the non-commutative derivative with respect to x_j. Prove Remark 3.3(2), i.e. prove that

$$p \otimes 1 - 1 \otimes p = \sum_{j=1}^{n} \big(\partial_j p \cdot x_j \otimes 1 - 1 \otimes x_j \cdot \partial_j p\big)$$

holds for any $p \in \mathbb{C}\langle x_1, \dots, x_n\rangle$.

Exercise 2. Let $\mathcal{B}$ be a unital algebra and consider the algebra $\mathcal{B}\langle x_1, \dots, x_n\rangle$ of non-commutative polynomials over $\mathcal{B}$ in n non-commuting (formal) variables $x_1, \dots, x_n$. For $j = 1, \dots, n$, we denote by

$$\partial_j : \ \mathcal{B}\langle x_1, \dots, x_n\rangle \to \mathcal{B}\langle x_1, \dots, x_n\rangle \otimes \mathcal{B}\langle x_1, \dots, x_n\rangle$$

the non-commutative derivative over $\mathcal{B}$ with respect to x_j.

(a) Let $\mathcal{M}$ be an arbitrary $\mathcal{B}\langle x_1, \dots, x_n\rangle$-bimodule and let

$$\delta : \ \mathcal{B}\langle x_1, \dots, x_n\rangle \to \mathcal{M}$$

be a non-commutative derivation over $\mathcal{B}$ in the sense that we have $\delta(b) = 0$ for all $b \in \mathcal{B}$ and

$$\delta(p_1 p_2) = p_1 \cdot \delta(p_2) + \delta(p_1) \cdot p_1 \qquad \text{for all } p_1, p_2 \in \mathcal{B}\langle x_1, \dots, x_n\rangle,$$

where $\cdot$ denotes the left and right action, respectively, of $\mathcal{B}\langle x_1, \dots, x_n\rangle$ on $\mathcal{M}$. Show that for any $p \in \mathcal{B}\langle x_1, \dots, x_n\rangle$

$$\delta(p) = \sum_{j=1}^{n} (\partial_j p)\sharp\delta(x_j),$$

where $\sharp : (\mathcal{B}\langle x_1, \dots, x_n\rangle \otimes \mathcal{B}\langle x_1, \dots, x_n\rangle) \times \mathcal{M} \to \mathcal{M}$ is defined by bilinear extension of $(p_1 \otimes p_2)\sharp m := p_1 \cdot m \cdot p_2$.

(b) Let $\mathcal{R}_\mathcal{B} \subseteq \mathcal{B}\langle x_1, \dots, x_n\rangle \otimes \mathcal{B}\langle x_1, \dots, x_n\rangle$ be the linear subspace spanned by

$$\{p_1 b \otimes p_2 - p_1 \otimes b p_2 |\ p_1, p_2 \in \mathcal{B}\langle x_1, \dots, x_n\rangle, b \in \mathcal{B}\}.$$

We consider the quotient space

$$\begin{aligned} \mathcal{M} &:= \mathcal{B}\langle x_1, \dots, x_n\rangle \otimes_\mathcal{B} \mathcal{B}\langle x_1, \dots, x_n\rangle \\ &:= (\mathcal{B}\langle x_1, \dots, x_n\rangle \otimes \mathcal{B}\langle x_1, \dots, x_n\rangle)/\mathcal{R}_\mathcal{B}, \end{aligned}$$

where we put $p_1 \otimes_\mathcal{B} p_2 := p_1 \otimes p_2 + \mathcal{R}_\mathcal{B}$. Justify that $\mathcal{M}$ is a $\mathcal{B}\langle x_1, \dots, x_n\rangle$-bimodule and show that $\delta : \mathcal{B}\langle x_1, \dots, x_n\rangle \to \mathcal{M}$ defined by

$$\delta(p) := p \otimes_\mathcal{B} 1 - 1 \otimes_\mathcal{B} p \qquad \text{for any } p \in \mathcal{B}\langle x_1, \dots, x_n\rangle$$

is a non-commutative derivation over $\mathcal{B}$.

(c) Apply the formula obtained in (a) to the derivation considered in (b). Convince yourself that your result reduces in the case $\mathcal{B} = \mathbb{C}$ to the formula that was proven in Problem 1.

(d) Consider $p \in \mathcal{B}\langle x_1, \dots, x_n\rangle$. Assume that $\partial_i p = 0$ for all $i = 1, \dots, n$. Show that $p \in \mathcal{B}$.

Exercise 3. Let $S_1, \dots, S_n$ be the n free semicircular elements from Example 1.7 (compare also Theorem 3.14). Fix a natural m. Let $f : \{1, \dots, n\}^m \to \mathbb{C}$ be any function that "vanishes on the diagonals", i.e., $f(i_1, \dots, i_m) = 0$ whenever there are $k \neq l$ such that $i_k = i_l$. Put

$$P := \sum_{i_1, \dots, i_m = 1}^{n} f(i_1, \dots, i_m) S_{i_1} \cdots S_{i_m} \in \mathbb{C}\langle S_1, \dots, S_n\rangle.$$

Calculate

$$\sum_{i=1}^{n} \partial_i^* \partial_i P.$$