Assignments for the lecture on Non-Commutative Distributions
Winter term 2014/2015

Assignment 2A
for the tutorial on Tuesday, 18 November (in SR6)

Exercise 1. The Catalan numbers \( C_n := \frac{1}{n+1} \binom{2n}{n} \) are characterized by the recursion formula:

\[
C_0 = C_1 = 1, \quad C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i}, \quad n \geq 2
\]

Let \( S := l(f) + l^*(f) \) be an element as in Example 1.7, for a unit vector \( f \in H \). Prove that it has the semi-circle distribution by showing:

\[
\varphi(S^k) = \begin{cases} 
0 & \text{if } k \text{ is odd} \\
C_n & \text{if } k = 2n
\end{cases}
\]

Exercise 2. Let \((\mathcal{A}, \varphi)\) and \((\mathcal{B}, \psi)\) be \(*\)-probability spaces (see Def. 1.1(2) of the book [NS] by Nica-Speicher) such that \( \varphi \) and \( \psi \) are faithful (see [NS, Def. 1.3]), i.e. \( \varphi(x^*x) = 0 \) implies \( x = 0 \). Let \( a_1, \ldots, a_n \in \mathcal{A} \) and \( b_1, \ldots, b_n \in \mathcal{B} \) and assume that the joint \( *\)-distributions of \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) coincide, i.e. for all monomials \( Q \in \mathbb{C}\langle z_1, z_1^*, z_2, z_2^*, \ldots, z_n, z_n^* \rangle \) we have:

\[
\varphi(Q(a_1, a_1^*, a_2, a_2^*, \ldots, a_n, a_n^*)) = \psi(Q(b_1, b_1^*, b_2, b_2^*, \ldots, b_n, b_n^*))
\]

(a) Assume that \( \mathcal{A} \) is generated by \( a_1, \ldots, a_n, 1_A \) as a \(*\)-algebra, and likewise \( \mathcal{B} \) with generators \( b_1, \ldots, b_n, 1_B \). Show that there is a \(*\)-isomorphism \( \mathcal{A} \to \mathcal{B} \) such that \( a_i \mapsto b_i \).

(b) Show that the same holds true, if \( \mathcal{A} \) and \( \mathcal{B} \) are \( C^*\)-algebras.

Exercise 3. Prove Remark 2.8: A function \( F \) respects intertwiners if for all \( z_1 \in M_n(\mathcal{B}), z_2 \in M_m(\mathcal{B}), T \in M_{n,m}(\mathbb{C}) \) the condition \( z_1 T = T z_2 \) implies \( F(z_1) T = T F(z_2) \). Show that a function is noncommutative if and only if it respects intertwiners.