UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 2A
for the tutorial on *Tuesday, 18 November* (in SR6)

Exercise 1. The *Catalan numbers* $C_n := \frac{1}{n+1}\binom{2n}{n}$ are characterized by the recursion formula:

$$C_0 = C_1 = 1, \qquad C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}, n \geq 2$$

Let $S := l(f) + l^*(f)$ be an element as in Example 1.7, for a unit vector $f \in H$. Prove that it has the *semi-circle distribution* by showing:

$$\varphi(S^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ C_n & \text{if } k = 2n \end{cases}$$

Exercise 2. Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be *-probability spaces (see Def. 1.1(2) of the book [NS] by Nica-Speicher) such that φ and ψ are *faithful* (see [NS, Def. 1.3]), i.e. $\varphi(x^*x) = 0$ implies $x = 0$. Let $a_1, \ldots, a_n \in \mathcal{A}$ and $b_1, \ldots, b_n \in \mathcal{B}$ and assume that the *joint *-distributions* of $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ coincide, i.e. for all monomials $Q \in \mathbb{C}\langle z_1, z_1^*, z_2, z_2^*, \ldots, z_n, z_n^*\rangle$ we have:

$$\varphi(Q(a_1, a_1^*, a_2, a_2^*, \ldots, a_n, a_n^*)) = \psi(Q(b_1, b_1^*, b_2, b_2^*, \ldots, b_n, b_n^*))$$

(a) Assume that $\mathcal{A}$ is generated by $a_1, \ldots, a_n, 1_{\mathcal{A}}$ as a *-algebra, and likewise $\mathcal{B}$ with generators $b_1, \ldots, b_n, 1_{\mathcal{B}}$. Show that there is a *-isomorphism $\mathcal{A} \to \mathcal{B}$ such that $a_i \mapsto b_i$.

(b) Show that the same holds true, if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

Exercise 3. Prove Remark 2.8: A function F *respects intertwiners* if for all $z_1 \in M_n(\mathcal{B})$, $z_2 \in M_m(\mathcal{B})$, $T \in M_{n,m}(\mathbb{C})$ the condition $z_1T = Tz_2$ implies $F(z_1)T = TF(z_2)$. Show that a function is noncommutative if and only if it repsects intertwiners.