UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 3A
for the tutorial on *Tuesday, 2 December* (in SR6)

Exercise 1. Let $(U_n)_{n\in\mathbb{N}_0}$ be the *(rescaled) Chebyshev polynomials of the second kind*, i.e. the sequence of polynomials $U_n \in \mathbb{C}\langle x\rangle$, which are defined recursively by

$$U_0(x) = 1, \quad U_1(x) = x, \qquad \text{and} \qquad U_{n+1}(x) = xU_n(x) - U_{n-1}(x) \quad \text{for all } n \in \mathbb{N}.$$

Let $\partial_x : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ be the non-commutative derivative with respect to x. Show that

$$\partial_x U_n(x) = \sum_{k=1}^{n} U_{k-1}(x) \otimes U_{n-k}(x) \qquad \text{for all } n \in \mathbb{N}.$$

Exercise 2. Let S be a semicircular variable as in Example 1.7 and let $\partial_S : \mathbb{C}\langle S\rangle \to \mathbb{C}\langle S\rangle \otimes \mathbb{C}\langle S\rangle$ be the non-commutative derivative with respect to S. Furthermore, we consider the polynomials $(U_n)_{n\in\mathbb{N}_0}$ from Exercise 1.
Prove the following statements:

(a) For all $m, n \in \mathbb{N}_0$, we have

$$\varphi(U_m(S)U_n(S)^*) = \delta_{m,n}.$$

Moreover, $U_n(S)^* = U_n(S)$.

(b) For any $P \in \mathbb{C}\langle S\rangle$, it holds true that

$$\|\partial_S^*(P \otimes 1)\|_2 = \|P\|_2$$

and

$$\|(\mathrm{id} \otimes \varphi)(\partial_S P)\|_2 \leq \|P\|_2.$$

Note that this is in fact a stronger version of Theorem 3.19.

(c) The statement in (b) shows, that $(\mathrm{id} \otimes \varphi) \circ \partial_S$ is a bounded operator (with respect to $\|\cdot\|_2$). Show that this is *not* true for ∂_S. To do so, show that $\|U_n\|_2 = 1$ and $\|\partial_S(U_n)\|_2 = \sqrt{n}$.