Exercise 1. Let \((U_n)_{n \in \mathbb{N}_0}\) be the (rescaled) Chebyshev polynomials of the second kind, i.e. the sequence of polynomials \(U_n \in \mathbb{C}(x)\), which are defined recursively by

\[
U_0(x) = 1, \quad U_1(x) = x, \quad \text{and} \quad U_{n+1}(x) = xU_n(x) - U_{n-1}(x) \quad \text{for all } n \in \mathbb{N}.
\]

Let \(\partial_x : \mathbb{C}(x) \to \mathbb{C}(x) \otimes \mathbb{C}(x)\) be the non-commutative derivative with respect to \(x\). Show that

\[
\partial_x U_n(x) = \sum_{k=1}^n U_k(x) \otimes U_{n-k}(x) \quad \text{for all } n \in \mathbb{N}.
\]

Exercise 2. Let \(S\) be a semicircular variable as in Example 1.7 and let \(\partial_S : \mathbb{C}(S) \to \mathbb{C}(S) \otimes \mathbb{C}(S)\) be the non-commutative derivative with respect to \(S\). Furthermore, we consider the polynomials \((U_n)_{n \in \mathbb{N}_0}\) from Exercise 1.

Prove the following statements:

(a) For all \(m, n \in \mathbb{N}_0\), we have

\[
\varphi(U_m(S)U_n(S)^*) = \delta_{m,n}.
\]

Moreover, \(U_n(S)^* = U_n(S)\).

(b) For any \(P \in \mathbb{C}(S)\), it holds true that

\[
\|\partial_S^* (P \otimes 1)\|_2 = \|P\|_2
\]

and

\[
\|(\text{id} \otimes \varphi)(\partial_S P)\|_2 \leq \|P\|_2.
\]

Note that this is in fact a stronger version of Theorem 3.19.

(c) The statement in (b) shows, that \((\text{id} \otimes \varphi) \circ \partial_S\) is a bounded operator (with respect to \(\| \cdot \|_2\)). Show that this is not true for \(\partial_S\). To do so, show that \(\|U_n\|_2 = 1\) and \(\|\partial_S(U_n)\|_2 = \sqrt{n}\).