UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 3B
for the tutorial on *Tuesday, 2 December* (in SR6)

Exercise 1. Let $\mathcal{H}$ be a Hilbert space. We consider the full Fock space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^{\otimes k}, \qquad \mathcal{H}^{\otimes 0} \cong \mathbb{C} \cdot \Omega$$

over $\mathcal{H}$, where Ω denotes the vacuum. As in Example 1.7 (see also Reminder 5.1), we put $S(f) := l(f) + l^*(f)$ for any vector $f \in \mathcal{H}$, where $l(f)$ and $l^*(f)$ are the creation and annihilation operator, respectively, with respect to f.
Now, let vectors $f_1, \ldots, f_k \in \mathcal{H}$ be given such that $\|f_j\| = 1$ for $j = 1, \ldots, k$ and

$$f_1 \perp f_2, \quad f_2 \perp f_3, \quad \ldots, \quad f_{k-1} \perp f_k.$$

Prove that

$$U_{n_1}(S(f_1))U_{n_2}(S(f_2)) \cdots U_{n_k}(S(f_k))\Omega = f_1^{\otimes n_1} \otimes f_2^{\otimes n_2} \otimes \cdots \otimes f_k^{\otimes n_k}$$

holds for all $n_1, \ldots, n_k \in \mathbb{N}$, where $(U_n)_{n \in \mathbb{N}_0}$ are the Chebyshev polynomials which we introduced in Exercise 1, Assignment 3A.

Exercise 2. Let $(S_t)_{t \geq 0}$ a free Brownian motion as it was introduced in Definition 5.4. Show that

$$\int_{[0,1]^n} dS_{t_1} \cdots dS_{t_n} = U_n\Big(\int_{[0,1]} dS_t\Big) \qquad \text{for all } n \in \mathbb{N},$$

where $(U_n)_{n \in \mathbb{N}_0}$ are again the Chebyshev polynomials which we introduced in Exercise 1, Assignment 3A.

Exercise 3. Consider the C^*-algebra $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$. By definition (cf. Section 4.4), we have

$$\mathbb{H}^+(M_n(\mathbb{C})) := \big\{ B \in M_n(\mathbb{C}) \big| \; \exists \varepsilon > 0 : \; \mathrm{Im}(B) \geq \varepsilon 1 \big\}, \qquad \mathrm{Im}(B) := \frac{1}{2i}(B - B^*).$$

In the case $n = 2$, show that in fact

$$\mathbb{H}^+(M_2(\mathbb{C})) = \left\{ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; \mathrm{Im}(b_{11}) > 0, \; \mathrm{Im}(b_{11}) \, \mathrm{Im}(b_{22}) > \frac{1}{4} \big| b_{12} - \overline{b_{21}} \big|^2 \right\}.$$

For general $n \in \mathbb{N}$, prove that if a matrix $B \in M_n(\mathbb{C})$ belongs to $\mathbb{H}^+(M_n(\mathbb{C}))$, then all eigenvalues of B lie in the complex upper half-plane $\mathbb{C}^+$. Is the converse also true?