UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 4A
for the tutorial on *Tuesday, 16 December* (in SR6)

Exercise 1. For any $n \in \mathbb{N}$ and each $f \in L^2(\mathbb{R}_+^n)$, we define $f^* \in L^2(\mathbb{R}_+^n)$ by

$$f^*(t_1, t_2, \dots, t_n) = \overline{f(t_n, \dots, t_2, t_1)}.$$

Show that the Wigner integrals $I_n(f), I_n(f^*) \in B(\mathcal{F}(L^2(\mathbb{R}_+))$ satisfy for all $f \in L^2(\mathbb{R}_+^n)$ the relation

$$I_n(f)^* = I_n(f^*).$$

Exercise 2. Let $f \in L^2(\mathbb{R}_+)$ and $g \in L^2(\mathbb{R}_+^2)$ be given. We define the *(first) contraction* $f \frown g \in L^2(\mathbb{R}_+)$ by

$$(f \frown g)(t_3) = \int_{\mathbb{R}_+} f(t) g(t, t_3)\, dt.$$

Prove that in this case the following version of *Itô's formula* holds true:

$$\int f(t_1)\, dS_{t_1} \cdot \int g(t_2, t_3)\, dS_{t_2} dS_{t_3} = \int f(t_1) g(t_2, t_3)\, dS_{t_1} dS_{t_2} dS_{t_3} + \int (f \frown g)(t_3)\, dS_{t_3}$$

Exercise 3. Let S be a semicircular variable as in Example 1.7 and let $\partial_S : \mathbb{C}\langle S \rangle \to \mathbb{C}\langle S \rangle \otimes \mathbb{C}\langle S \rangle$ be the non-commutative derivative with respect to S, which we regard as an unbounded linear operator

$$\partial_S : \; L^2(S; \varphi) \supset D(\partial_S) \to L^2(S; \varphi) \otimes L^2(S; \varphi)$$

with domain $D(\partial_S) = \mathbb{C}\langle S \rangle$. Furthermore, we consider the rescaled Chebyshev polynomials of the second kind $(U_n)_{n \in \mathbb{N}_0}$, which we introduced in Exercise 1, Assignment 3A. Show that for all $n, m \in \mathbb{N}_0$

$$\partial_S^*(U_n(S) \otimes U_m(S)) = U_{n+m+1}(S).$$