UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
Prof. Dr. Roland Speicher
Dr. Moritz Weber

Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 4B
for the tutorial on *Tuesday, 16 December* (in SR6)

Exercise 1. We consider the non-commutative C^*-probability space $(M_n(\mathbb{C}), \mathrm{tr}_n)$, which is obtained by endowing the C^*-algebra $M_n(\mathbb{C})$ of all complex $n \times n$ matrices with the normalized trace $\mathrm{tr}_n : M_n(\mathbb{C}) \to \mathbb{C}$ defined by

$$\mathrm{tr}_n(A) := \frac{1}{n} \sum_{i=1}^{n} a_{i,i} \qquad \text{for any matrix } A = (a_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{C}).$$

Show that

$$\lim_{k \to \infty} \Big(\mathrm{tr}_n \big((A^*A)^k \big) \Big)^{\frac{1}{2k}} = \|A\| \qquad \text{for all } A \in M_n(\mathbb{C}).$$

Hint: Basic linear algebra might be helpful.

Exercise 2. Let (M, τ) be a tracial W^*-probability space and let self-adjoint elements $X_1, \dots, X_n \in M$ be given. We assume that a conjugate system $(\xi_1, \dots, \xi_n)$ for $(X_1, \dots, X_n)$ exists.

Prove Lemma 6.5: For all $P_1, P_2 \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ and for each $j = 1, \dots, n$, we have the following inequalities:

(a)
$$\|\partial_j^*(P_1 \otimes P_2)\|_2 \leq 3 \|\xi_j\|_2 \|P_1\| \|P_2\|$$

(b)
$$\|(\mathrm{id} \otimes \tau)((\partial_j P_1) \cdot P_2)\|_2 \leq 4 \|\xi_j\|_2 \|P_1\| \|P_2\| \qquad \text{and}$$
$$\|(\tau \otimes \mathrm{id})(P_1 \cdot (\partial_j P_2))\|_2 \leq 4 \|\xi_j\|_2 \|P_1\| \|P_2\|$$

Hint: Combine Theorem 3.9 and Theorem 3.19 in order to prove (a). For proving (b), use in addition the idea of "integration by parts".