UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 5A
for the tutorial on *Tuesday, 20 January* (in SR6)

These excercises will follow excercise 3.2 in Vern Paulsen's book, *Completely Bounded Maps and Operator Algebras*. You may visit Williams if you would like to see his copy of this book.

Exercise 1. Let P, Q, A be operators on some Hilbert space $\mathcal{H}$ and assume that P and Q are positive. The identity operator on $\mathcal{H}$ will be denoted by 1.

(a) Show that $\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0$ if and only if

$$|\langle Ax, y\rangle|^2 \leq \langle Py, y\rangle\langle Qx, x\rangle \qquad \text{for all } x, y \in \mathcal{H}.$$

Hint: this may be reduced to Lemma 3.1 (i) in Paulsen's book (which guarantees the validity of (a) in the special case $P = Q = 1$) but the proof is distasteful. Extra points if you can provide a proof based on more basic principles.

(b) Prove Lemma 3.1. (ii) of Vern Paulsen's book: $\begin{pmatrix} 1 & A \\ A^* & Q \end{pmatrix} \geq 0$ if and only if $A^*A \leq Q$.

(c) Show that if $\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \geq 0$, then for any $x \in \mathcal{H}$ we have that

$$0 \leq \langle (P + A + A^* + Q)x, x\rangle \leq \left(\sqrt{\langle Px, x\rangle} + \sqrt{\langle Qx, x\rangle}\right)^2$$

and hence

$$\|P + A + A^* + Q\| \leq (\|P\|^{1/2} + \|Q\|^{1/2})^2.$$

(d) Show that if $\begin{pmatrix} P & A \\ A^* & P \end{pmatrix} \geq 0$, then $A^*A \leq \|P\|P$ and, in particular, $\|A\| \leq \|P\|$.