Exercise 1. Let \((G, \cdot)\) be a group. A function \(F : G \to \mathbb{C}\) is called positive definite on \(G\), if the condition
\[
\sum_{g_1, g_2 \in G} F(g_2^{-1} \cdot g_1) f(g_1) \overline{f(g_2)} \geq 0
\]
is satisfied for each function \(f : G \to \mathbb{C}\) with finite support, i.e. \(\#\{g \in G \mid f(g) \neq 0\} < \infty\). Prove the following statements:

(a) Let \(F_1, F_2 : G \to \mathbb{C}\) be positive definite functions on \(G\), then their pointwise product
\[
F : G \to \mathbb{C}, \ g \mapsto F_1(g)F_2(g)
\]
is also a positive definite function on \(G\).

(b) If we consider the group \((\mathbb{R}, +)\), then for any \(\omega \in (0, \infty)\), the function
\[
F : \mathbb{R} \to \mathbb{C}, \ x \mapsto e^{-\omega x^2}
\]
is positive definite on \(\mathbb{R}\).

Exercise 2. For \(n \in \mathbb{N}\), let \(\mathfrak{S}_n\) denote the group of all permutations of \(\{1, \ldots, n\}\). As in the Definition 9.1 of the lecture, we denote by \(i(\pi)\) the number of inversions of any permutation \(\pi \in \mathfrak{S}_n\), i.e.
\[
i(\pi) := \#\{(k, l) \mid 1 \leq k < l \leq n, \ \pi(k) > \pi(l)\}.
\]

(a) For \(1 \leq k \leq n\), we consider the permutation \(T_{k,n} \in \mathfrak{S}_n\) that is given by
\[
T_{k,n} := \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & k-1 & n & k & \cdots & n-2 & n-1 \end{pmatrix}.
\]
Calculate \(i(T_{k,n})\) and \(T_{k,n}^{-1}\).

(b) Prove that each permutation \(\pi \in \mathfrak{S}_n\) can be represented uniquely as \(\pi = \sigma T_{k,n}\) with \(\sigma \in \mathfrak{S}_{n-1}\) and \(1 \leq k \leq n\), where we identify \(\mathfrak{S}_{n-1}\) with the subgroup of \(\mathfrak{S}_n\) consisting of all permutations fixing \(n\).

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(c) In the situation of (b), we have that $i(\pi) = i(\sigma) + i(T_{k,n})$.

(d) In the corresponding group algebra

$$
\mathbb{C}[\mathfrak{S}_n] = \left\{ \sum_{\pi \in \mathfrak{S}_n} a_\pi \pi \mid a_\pi \in \mathbb{C} \text{ for all } \pi \in \mathfrak{S}_n \right\},
$$
on which addition and multiplication is defined by

$$
\left( \sum_{\pi \in \mathfrak{S}_n} a_\pi \pi \right) + \left( \sum_{\sigma \in \mathfrak{S}_n} b_\sigma \sigma \right) := \sum_{\tau \in \mathfrak{S}_n} (a_\tau + b_\tau) \tau
$$

and

$$
\left( \sum_{\pi \in \mathfrak{S}_n} a_\pi \pi \right) \cdot \left( \sum_{\sigma \in \mathfrak{S}_n} b_\sigma \sigma \right) := \sum_{\pi, \sigma \in \mathfrak{S}_n} (a_\pi b_\sigma) (\pi \sigma) = \sum_{\tau \in \mathfrak{S}_n} \left( \sum_{\pi, \sigma \in \mathfrak{S}_n \tau = \pi \sigma} a_\pi b_\sigma \right) \tau,
$$

respectively, we consider for any fixed $-1 \le q \le 1$ the elements

$$
\alpha_n := \sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} \pi \quad \text{and} \quad \beta_n := \sum_{k=1}^{n} q^{i(T_{k,n})} T_{k,n}.
$$

Show that $\alpha_n = \alpha_{n-1} \beta_n$.

**Remark:** In the proof of Theorem 9.4, which was given in the lecture, we observed that

$$
\langle \eta, \eta \rangle_q = \sum_{\pi \in \mathfrak{S}_n} q^{i(\pi)} \langle \eta, \pi \eta \rangle_{\mathcal{H}^\otimes n} \quad \text{for any } \eta \in \mathcal{H}^\otimes n,
$$

where $\pi \in \mathfrak{S}_n$ acts on the algebraic tensor product $\mathcal{H}^\otimes n$ in the obvious way by permuting the factors, i.e., if we have $\eta = \eta_1 \otimes \cdots \otimes \eta_n$, then $\pi \eta = \eta_{\pi(1)} \otimes \cdots \otimes \eta_{\pi(n)}$. Clearly, this action of $\mathfrak{S}_n$ on $\mathcal{H}^\otimes n$ extends to an action of $\mathbb{C}[\mathfrak{S}_n]$ on $\mathcal{H}^\otimes n$. Hence, by using the notation from above, we can write

$$
\langle \eta, \eta \rangle_q = \langle \eta, \alpha_n \eta \rangle_{\mathcal{H}^\otimes n} \quad \text{for all } \eta \in \mathcal{H}^\otimes n,
$$

so that the question whether $\langle \cdot, \cdot \rangle_q$ is positive definite amounts to the question of positivity of the operator $\alpha_n$ on $\mathcal{H}^\otimes n$. This observation is at the base of D. Zagier’s proof of Theorem 9.4, which can be found in his paper *Realizability of a model in infinite statistics (1992)*; and the formula in (d) is one of the crucial steps thereof.

**Exercise 3.** Prove the following statements:

(a) For all $n \in \mathbb{N}$, we have

$$
\sum_{\pi \in \mathfrak{S}_n} (-1)^{i(\pi)} = \begin{cases} 
1, & \text{if } n = 1 \\
0, & \text{otherwise}
\end{cases},
$$

where $i(\pi)$ denotes the number of *inversions* of any permutation $\pi \in \mathfrak{S}_n$ as it was defined in Definition 9.1 of the lecture (and also recalled in Exercise 2).

(b) For all $m \in \mathbb{N}$, it holds true that

$$
\sum_{\pi \in \mathcal{P}_2(2m)} (-1)^{i(\pi)} = 1,
$$

where $i(\pi)$ denotes the number of *crossings* of any pairing $\pi \in \mathcal{P}_2(2m)$ as it was defined in Theorem 9.7 of the lecture.