UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 5B
for the tutorial on *Tuesday, 20 January* (in SR6)

Exercise 1. Let $(G, \cdot)$ be a group. A function $F : G \to \mathbb{C}$ is called *positive definite on* G, if the condition

$$\sum_{g_1, g_2 \in G} F(g_2^{-1} \cdot g_1)\, f(g_1)\overline{f(g_2)} \geq 0$$

is satisfied for each function $f : G \to \mathbb{C}$ with finite support, i.e. $\#\{g \in G \mid f(g) \neq 0\} < \infty$. Prove the following statements:

(a) Let $F_1, F_2 : G \to \mathbb{C}$ be positive definite functions on G, then their pointwise product

$$F :\ G \to \mathbb{C},\ g \mapsto F_1(g)F_2(g)$$

is also a positive definite function on G.

(b) If we consider the group $(\mathbb{R}, +)$, then for any $\omega \in (0, \infty)$, the function

$$F :\ \mathbb{R} \to \mathbb{C},\ x \mapsto e^{-\omega x^2}$$

is positive definite on $\mathbb{R}$.

Exercise 2. For $n \in \mathbb{N}$, let $\mathfrak{S}_n$ denote the group of all permutations of $\{1, \ldots, n\}$. As in the Definition 9.1 of the lecture, we denote by $i(\pi)$ the number of *inversions* of any permutation $\pi \in \mathfrak{S}_n$, i.e.

$$i(\pi) := \#\{(k, l) \mid 1 \leq k < l \leq n,\ \pi(k) > \pi(l)\}.$$

(a) For $1 \leq k \leq n$, we consider the permutation $T_{k,n} \in \mathfrak{S}_n$ that is given by

$$T_{k,n} := \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n-1 & n \\ 1 & 2 & \cdots & k-1 & n & k & \cdots & n-2 & n-1 \end{pmatrix}.$$

Calculate $i(T_{k,n})$ and $T_{k,n}^{-1}$.

(b) Prove that each permutation $\pi \in \mathfrak{S}_n$ can be represented uniquely as $\pi = \sigma T_{k,n}$ with $\sigma \in \mathfrak{S}_{n-1}$ and $1 \leq k \leq n$, where we identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ consisting of all permutations fixing n.

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(c) In the situation of (b), we have that $i(\pi) = i(\sigma) + i(T_{k,n})$.

(d) In the corresponding group algebra

$$\mathbb{C}[\mathfrak{S}_n] = \Big\{ \sum_{\pi\in\mathfrak{S}_n} a_\pi\, \pi \Big|\, a_\pi \in \mathbb{C} \text{ for all } \pi \in \mathfrak{S}_n \Big\},$$

on which addition and multiplication is defined by

$$\Big(\sum_{\pi\in\mathfrak{S}_n} a_\pi\, \pi\Big) + \Big(\sum_{\sigma\in\mathfrak{S}_n} b_\sigma\, \sigma\Big) := \sum_{\tau\in\mathfrak{S}_n} (a_\tau + b_\tau)\, \tau$$

and

$$\Big(\sum_{\pi\in\mathfrak{S}_n} a_\pi\, \pi\Big) \cdot \Big(\sum_{\sigma\in\mathfrak{S}_n} b_\sigma\, \sigma\Big) := \sum_{\pi,\sigma\in\mathfrak{S}_n} (a_\pi b_\sigma)\,(\pi\sigma) = \sum_{\tau\in\mathfrak{S}_n} \Big(\sum_{\substack{\pi,\sigma\in\mathfrak{S}_n\\ \tau=\pi\sigma}} a_\pi b_\sigma \Big)\, \tau,$$

respectively, we consider for any fixed $-1 \leq q \leq 1$ the elements

$$\alpha_n := \sum_{\pi\in\mathfrak{S}_n} q^{i(\pi)}\, \pi \qquad \text{and} \qquad \beta_n := \sum_{k=1}^{n} q^{i(T_{k,n})}\, T_{k,n}.$$

Show that $\alpha_n = \alpha_{n-1}\beta_n$.

Remark: In the proof of Theorem 9.4, which was given in the lecture, we observed that

$$\langle \eta, \eta \rangle_q = \sum_{\pi\in\mathfrak{S}_n} q^{i(\pi)}\, \langle \eta, \pi\eta \rangle_{\mathcal{H}^{\otimes n}} \qquad \text{for any } \eta \in \mathcal{H}^{\otimes n},$$

where $\pi \in \mathfrak{S}_n$ acts on the algebraic tensor product $\mathcal{H}^{\otimes n}$ in the obvious way by permuting the factors, i.e., if we have $\eta = \eta_1 \otimes \dots \otimes \eta_n$, then $\pi\eta = \eta_{\pi(1)} \otimes \dots \otimes \eta_{\pi(n)}$. Clearly, this action of $\mathfrak{S}_n$ on $\mathcal{H}^{\otimes n}$ extends to an action of $\mathbb{C}[\mathfrak{S}_n]$ on $\mathcal{H}^{\otimes n}$. Hence, by using the notation from above, we can write

$$\langle \eta, \eta \rangle_q = \langle \eta, \alpha_n \eta \rangle_{\mathcal{H}^{\otimes n}} \qquad \text{for all } \eta \in \mathcal{H}^{\otimes n},$$

so that the question whether $\langle \cdot, \cdot \rangle_q$ is positive definite amounts to the question of positivity of the operator α_n on $\mathcal{H}^{\otimes n}$. This observation is at the base of D. Zagier's proof of Theorem 9.4, which can be found in his paper *Realizability of a model in infinite statistics (1992)*, and the formula in (d) is one of the crucial steps thereof.

Exercise 3. Prove the following statements:

(a) For all $n \in \mathbb{N}$, we have

$$\sum_{\pi\in\mathfrak{S}_n} (-1)^{i(\pi)} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases},$$

where $i(\pi)$ denotes the number of *inversions* of any *permutation* $\pi \in \mathfrak{S}_n$ as it was defined in Definition 9.1 of the lecture (and also recalled in Exercise 2).

(b) For all $m \in \mathbb{N}$, it holds true that

$$\sum_{\pi\in\mathcal{P}_2(2m)} (-1)^{i(\pi)} = 1,$$

where $i(\pi)$ denotes the number of *crossings* of any *pairing* $\pi \in \mathcal{P}_2(2m)$ as it was defined in Theorem 9.7 of the lecture.