Exercise 1. Let \((\mathcal{A}, \mathcal{E}, \mathcal{B})\) be an operator-valued \(C^\ast\)-probability space and let
\[
\eta = (\eta_1, \eta_2, \eta_3, \ldots)
\]
be a sequence of (completely positive) linear maps \(\eta_n : \mathcal{B} \to \mathcal{B}\). For any \(\pi \in \text{NC}_2(2n)\) with \(n \in \mathbb{N}\), we may define a mapping \(\eta_\pi : \mathcal{B}^{2n-1} \to \mathcal{B}\) recursively by the following rules:

- Let \(d\) be the depth of \(\pi\), i.e. the maximal depth of all blocks in \(\pi\). Let \((k, k+1)\) with \(1 \leq k \leq 2n-1\) be any block of maximal depth \(d\) and denote by \(\pi' \in \text{NC}_2(2(n-1))\) the non-crossing partition that is obtained by removing the block \((k, k+1)\) from \(\pi\). Then we put
  \[
  \eta_\pi[b_1, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_{2n-1}] := \eta_{\pi'}[b_1, \ldots, b_{k-1}, \eta_d(b_k)b_{k+1}, \ldots, b_{2n-1}]
  \]
  for all \(b_1, \ldots, b_{2n-1} \in \mathcal{B}\).

- If \(\pi \in \text{NC}_2(2)\), then we put \(\eta_\pi[b_1] := \eta_1(b_1)\) for all \(b_1 \in \mathcal{B}\).

Next, we define for any \(n \in \mathbb{N}\) the \(2n\)-th moment mapping \(m_{2n}^\eta : \mathcal{B}^{2n-1} \to \mathcal{B}\) by
\[
m_{2n}^\eta(b_1, b_2, \ldots, b_{2n-1}) := \sum_{\pi \in \text{NC}_2(2n)} \eta_\pi[b_1, b_2, \ldots, b_{2n-1}]
\]
for all \(b_1, b_2, \ldots, b_{2n-1} \in \mathcal{B}\).

Show that the series
\[
G^\eta(b) := b^{-1} + \sum_{n=1}^{\infty} m_{2n}^\eta(b^{-1}, b^{-1}, \ldots, b^{-1})
\]
(of which we assume that it converges on \(\{b \in \mathcal{B} | b\text{ invertible with }\|b^{-1}\| < r\}\) for some \(r > 0\)) has an operator-valued continued fraction expansion
\[
G^\eta(b) = \frac{1}{b - \eta_1\left(\frac{1}{b - \eta_2\left(\frac{1}{b - \ldots}\right)}\right)}.
\]

For doing so, show that
\[
G^\eta(b) = \frac{1}{b - \eta_1(G^\eta(b))},
\]
where we put \(\tilde{\eta} := (\eta_2, \eta_3, \eta_4, \ldots)\).

please turn the page
**Exercise 2.** For $0 < q < 1$, we consider the $q$-Fock space $F_q(\mathcal{H})$ over some Hilbert space $\mathcal{H}$. Let $f_1, f_2 \in \mathcal{H}$ with $f_1 \perp f_2$ be given and put

$$X := \begin{pmatrix} X(f_1) & 0 \\ 0 & X(f_2) \end{pmatrix}.$$ 

Show that there is no sequence $\eta = (\eta_1, \eta_2, \eta_3, \ldots)$ of linear maps $\eta_n : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, such that (with the notation from Exercise 1)

$$E[Xb_1 Xb_2 \cdots Xb_{2n-1} X] = m_{2n}^q(b_1, b_2, \ldots, b_{2n-1})$$

holds for all $n \in \mathbb{N}$ and all $b_1, b_2, \ldots, b_{2n-1} \in M_2(\mathbb{C})$, where the conditional expectation $E$ is given by

$$E \left[ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \right] = \left( \begin{pmatrix} \langle X_{11}\Omega, \Omega \rangle_q & \langle X_{12}\Omega, \Omega \rangle_q \\ \langle X_{21}\Omega, \Omega \rangle_q & \langle X_{22}\Omega, \Omega \rangle_q \end{pmatrix} \right).$$

**Hint:** Examine the second moment ($n = 1$) and the fourth moment ($n = 2$) of $X$.

**Exercise 3.** On $\mathbb{C}\langle x_1, \ldots, x_n \rangle$, we consider the cyclic derivatives

$$D_j : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle, \quad j = 1, \ldots, n,$$

as they were defined in Definition 10.3 of the lecture. Moreover, we denote by $\sigma$ the flip mapping

$$\sigma : \mathbb{C}\langle x_1, \ldots, x_n \rangle \otimes^2 \to \mathbb{C}\langle x_1, \ldots, x_n \rangle \otimes^2,$$

which is determined by linear extension of $\sigma(q_1 \otimes q_2) = q_2 \otimes q_1$. Furthermore, recall that the operation $\sharp$ is defined by $(q_1 \otimes q_2) \sharp p = q_1 p q_2$.

(a) Show that for $j = 1, \ldots, n$ the following kind of product rule

$$D_j(q_1 q_2) = \sigma(\partial_j q_1) \sharp q_2 + \sigma(\partial_j q_2) \sharp q_1$$

holds true for arbitrary $q_1, q_2 \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$.

(b) Let non-commutative polynomials $p_1, \ldots, p_n \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be given. Prove that the following statements are equivalent:

(i) $(p_1, \ldots, p_n)$ is a cyclic gradient, i.e. there exists a polynomial $p \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$, such that

$$p_j = D_j p \quad \text{for all } j = 1, \ldots, n.$$

(ii) $(p_1, \ldots, p_n)$ satisfies the integrability condition

$$\partial_j p_i = \sigma(\partial_i p_j) \quad \text{for all } i, j = 1, \ldots, n.$$