UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 6A
for the tutorial on *Tuesday, 3 February* (in SR6)

Exercise 1. Let $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ be an operator-valued C^*-probability space and let

$$\eta = (\eta_1, \eta_2, \eta_3, \dots)$$

be a sequence of (completely positive) linear maps $\eta_n : \mathcal{B} \to \mathcal{B}$. For any $\pi \in \mathrm{NC}_2(2n)$ with $n \in \mathbb{N}$, we may define a mapping $\eta_\pi : \ \mathcal{B}^{2n-1} \to \mathcal{B}$ recursively by the following rules:

- Let d be the depth of π, i.e. the maximal depth of all blocks in π. Let $(k, k+1)$ with $1 \leq k \leq 2n-1$ be any block of maximal depth d and denote by $\pi' \in \mathrm{NC}_2(2(n-1))$ the non-crossing partition that is obtained by removing the block $(k, k+1)$ from π. Then we put
$$\eta_\pi[b_1, \dots, b_{k-1}, b_k, b_{k+1}, \dots, b_{2n-1}] := \eta_{\pi'}[b_1, \dots, b_{k-1}\eta_d(b_k)b_{k+1}, \dots, b_{2n-1}]$$
for all $b_1, \dots, b_{2n-1} \in \mathcal{B}$.
- If $\pi \in \mathrm{NC}_2(2)$, then we put $\eta_\pi[b_1] := \eta_1(b_1)$ for all $b_1 \in \mathcal{B}$.

Next, we define for any $n \in \mathbb{N}$ the $2n$*-th moment mapping* $m_{2n}^\eta : \mathcal{B}^{2n-1} \to \mathcal{B}$ by

$$m_{2n}^\eta(b_1, b_2, \dots, b_{2n-1}) := \sum_{\pi \in \mathrm{NC}_2(2n)} \eta_\pi[b_1, b_2 \dots, b_{2n-1}] \qquad \text{for all } b_1, b_2, \dots, b_{2n-1} \in \mathcal{B}.$$

Show that the series

$$G^\eta(b) := b^{-1} + \sum_{n=1}^{\infty} m_{2n}^\eta(b^{-1}, b^{-1}, \dots, b^{-1})$$

(of which we assume that it converges on $\{b \in \mathcal{B} |\ b \text{ invertible with } \|b^{-1}\| < r\}$ for some $r > 0$) has an operator-valued continued fraction expansion

$$G^\eta(b) = \cfrac{1}{b - \eta_1\left(\cfrac{1}{b - \eta_2\left(\cfrac{1}{b - \ddots}\right)}\right)}.$$

For doing so, show that

$$G^\eta(b) = \frac{1}{b - \eta_1(G^{\tilde{\eta}}(b))},$$

where we put $\tilde{\eta} := (\eta_2, \eta_3, \eta_4, \dots)$.

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Exercise 2. For $0 < q < 1$, we consider the q-Fock space $\mathcal{F}_q(\mathcal{H})$ over some Hilbert space $\mathcal{H}$. Let $f_1, f_2 \in \mathcal{H}$ with $f_1 \perp f_2$ be given and put

$$X := \begin{pmatrix} X(f_1) & 0 \\ 0 & X(f_2) \end{pmatrix}.$$

Show that there is *no* sequence $\eta = (\eta_1, \eta_2, \eta_3, \dots)$ of linear maps $\eta_n : M_2(\mathbb{C}) \to M_2(\mathbb{C})$, such that (with the notation from Exercise 1)

$$\mathbb{E}[X b_1 X b_2 X \cdots X b_{2n-1} X] = m^{\eta}_{2n}(b_1, b_2, \dots, b_{2n-1})$$

holds for all $n \in \mathbb{N}$ and all $b_1, b_2, \dots, b_{2n-1} \in M_2(\mathbb{C})$, where the conditional expectation $\mathbb{E}$ is given by

$$\mathbb{E}\left[\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\right] = \begin{pmatrix} \langle X_{11}\Omega, \Omega\rangle_q & \langle X_{12}\Omega, \Omega\rangle_q \\ \langle X_{21}\Omega, \Omega\rangle_q & \langle X_{22}\Omega, \Omega\rangle_q \end{pmatrix}.$$

Hint: Examine the second moment ($n = 1$) and the fourth moment ($n = 2$) of X.

Exercise 3. On $\mathbb{C}\langle x_1, \dots, x_n\rangle$, we consider the cyclic derivatives

$$\mathcal{D}_j : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle, \qquad j = 1, \dots, n,$$

as they were defined in Definition 10.3 of the lecture. Moreover, we denote by σ the *flip mapping*

$$\sigma : \ \mathbb{C}\langle x_1, \dots, x_n\rangle^{\otimes 2} \to \mathbb{C}\langle x_1, \dots, x_n\rangle^{\otimes 2},$$

which is determined by linear extension of $\sigma(q_1 \otimes q_2) = q_2 \otimes q_1$. Furthermore, recall that the operation $\sharp$ is defined by $(q_1 \otimes q_2)\sharp p = q_1 p q_2$.

(a) Show that for $j = 1, \dots, n$ the following kind of product rule

$$\mathcal{D}_j(q_1 q_2) = \sigma(\partial_j q_1)\sharp q_2 + \sigma(\partial_j q_2)\sharp q_1$$

holds true for arbitrary $q_1, q_2 \in \mathbb{C}\langle x_1, \dots, x_n\rangle$.

(b) Let non-commutative polynomials $p_1, \dots, p_n \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be given. Prove that the following statements are equivalent:

(i) $(p_1, \dots, p_n)$ is a cyclic gradient, i.e. there exists a polynomial $p \in \mathbb{C}\langle x_1, \dots, x_n\rangle$, such that

$$p_j = \mathcal{D}_j p \qquad \text{for all } j = 1, \dots, n.$$

(ii) $(p_1, \dots, p_n)$ satisfies the integrability condition

$$\partial_j p_i = \sigma\big(\partial_i p_j\big) \qquad \text{for all } i, j = 1, \dots, n.$$