UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on *Non-Commutative Distributions*
Winter term 2014/2015

Assignment 6B
for the tutorial on *Tuesday, 3 February* (in SR6)

Exercise 1. In Definition 8.10, we extended the non-commutative derivatives as

$$\partial_j : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!) \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^{\otimes 2}, \qquad j=1,\ldots,n,$$

to the universal skew field of non-commutative fractions $\mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)$. Therefore, we also get an extension of the cyclic derivatives,

$$\mathcal{D}_j : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!) \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!), \qquad j=1,\ldots,n,$$

by putting $\mathcal{D}_j := m\circ\sigma\circ\partial_j$ for $j=1,\ldots,n$, where

$$\sigma : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^{\otimes 2} \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^{\otimes 2}$$

denotes the flip mapping given by $\sigma(r_1\otimes r_2) = r_2\otimes r_1$ and

$$m : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^{\otimes 2} \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)$$

denotes the multiplication map defined by $m(r_1\otimes r_2) = r_1 r_2$. As before, we introduce the cyclic gradient by

$$\mathcal{D} : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!) \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^n,\ r\mapsto (\mathcal{D}_1 r,\ldots,\mathcal{D}_n r).$$

(a) Prove that for each $r\in\mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)$ the formula

$$[r, 1\otimes 1] = \sum_{j=1}^n (\partial_j r)\sharp[x_j, 1\otimes 1]$$

holds true, where we extend the operation $\sharp$ in the obvious way.

(b) For $j=1,\ldots,n$, we put $\tilde{\partial}_j := \sigma\circ\partial_j$. Show that

$$[x_j, \mathcal{D}_j r] = m\big([x_j, \tilde{\partial}_j r]\big) \qquad\text{and}\qquad [x_j, \tilde{\partial}_j r] = -\sigma\big((\partial_j r)\sharp[x_j, 1\otimes 1]\big)$$

holds for all $r\in\mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)$ and $j=1,\ldots,n$.

(c) Consider the mapping

$$\Theta : \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)^n \to \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!),\ (r_1,\ldots,r_n)\mapsto \sum_{j=1}^n [x_j, r_j].$$

Prove that

$$\mathbb{C}1 + [\mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!), \mathbb{C}(\!\langle x_1,\ldots,x_n\rangle\!)] \subseteq \ker(\mathcal{D}) \qquad\text{and}\qquad \operatorname{ran}(\mathcal{D}) \subseteq \ker(\Theta).$$

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Exercise 2. Let a positive definite matrix $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ be given and consider the non-commutative polynomial $q \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ that is defined by

$$q(x_1, \dots, x_n) := \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}\, x_i x_j.$$

Show that there is a unique Gibbs state τ_q on $\mathbb{C}\langle x_1, \dots, x_n\rangle$ for the potential q and that we can find moreover an isomorphism $\Phi : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle$, induced by a linear change of variables, such that

$$\tau_q = \mu_{S_1,\dots,S_n} \circ \Phi,$$

where $\mu_{S_1,\dots,S_n}$ denotes the semicircular distribution $\mathbb{C}\langle x_1, \dots, x_n\rangle$ as it was introduced in Example 1.8 and recovered in Example 11.4 (1).