6. Algebraic relations and zero divisors

6.1. Reminder:

Let (M,τ) be a tracial W^*-probability space and let $X_j = X_j^* \in M$, $j=1,\dots,n$, be given.

If $X_1,\dots,X_n$ do not satisfy any algebraic relation, i.e.

$$P(X_1,\dots,X_n) \neq 0 \qquad \forall\, 0 \neq P \in \mathbb{C}\langle X_1,\dots,X_n\rangle,$$

then the non-commutative derivatives

$$\partial_{X_j} : \mathbb{C}\langle X_1,\dots,X_n\rangle \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}$$

with $\partial_{X_j} X_i = \delta_{i,j}\, 1 \otimes 1$ are well-defined.

If there are $\xi_1,\dots,\xi_n \in L^2(X_1,\dots,X_n;\tau)$ such that for all $P \in \mathbb{C}\langle X_1,\dots,X_n\rangle$

$$(*) \quad \tau\otimes\tau(\partial_{X_j} P) = \tau(P\xi_j), \quad j=1,\dots,n$$

holds, we call $(\xi_1,\dots,\xi_n)$ the <u>conjugate system</u> for $(X_1,\dots,X_n)$. Note that in this case $1\otimes 1 \in D(\partial_{X_j}^*)$ and

$$\xi_j = \partial_{X_j}^*(1\otimes 1), \quad j=1,\dots,n,$$

where we regard ∂_{X_j} as an unbounded linear operator

$$\partial_{X_j} : L^2(X_1,\dots,X_n;\tau) \supseteq D(\partial_{X_j}) \longrightarrow L^2(X_1,\dots,X_n;\tau)^{\otimes 2}$$

with domain $D(\partial_{X_j}) := \mathbb{C}\langle X_1,\dots,X_n\rangle$.

We define the <u>Fisher information</u> by

$$\Phi^*(X_1,\dots,X_n) := \sum_{j=1}^{n} \|\xi_j\|^2_{L^2(X_1,\dots,X_n;\tau)}.$$

<u>Question:</u>

Do we really need the assumption that $X_1,\dots,X_n$ do not satisfy any (non-trivial) algebraic relation?

NO, this is in fact a consequence of (*)!

We just have to replace ∂_{X_j} by

$$\partial_j : \mathbb{C}\langle x_1,\dots,x_n\rangle \longrightarrow \mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$$

and use evaluation at $X = (X_1,\dots,X_n)$:

$$ev_X : \mathbb{C}\langle x_1,\dots,x_n\rangle \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle, \quad x_j \longmapsto X_j$$

$$ev_X \otimes ev_X : \mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2} \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}.$$

6.2. Theorem:

Let (M, τ) be a tracial W^*-probability space and let $X_j = X_j^* \in M$, $j = 1, \dots, n$, be given. Assume that there are $\xi_1, \dots, \xi_n \in L^2(X_1, \dots, X_n; \tau)$ such that for all $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, $j = 1, \dots, n$

$$\tau \otimes \tau \big((\partial_j P)(X_1, \dots, X_n) \big) = \tau \big(P(X_1, \dots, X_n)\, \xi_j \big)$$

holds, then we have:

(a) $X_1, \dots, X_n$ do not satisfy any (non-trivial) algebraic relation.

(b) For $j = 1, \dots, n$, there is a derivation
$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \longrightarrow \mathbb{C}\langle X_1, \dots, X_n \rangle^{\otimes 2}$$
such that $\hat{\partial}_j X_i = \delta_{i,j}\, 1 \otimes 1$, $i = 1, \dots, n$.

Proof:

① We consider the ideals

$$I_X^1 := \{ P \in \mathbb{C}\langle X_1, \dots, X_n \rangle \mid P(X_1, \dots, X_n) = 0 \}$$

and

$$I_X^2 := \{ Q \in \mathbb{C}\langle X_1, \dots, X_n \rangle^{\otimes 2} \mid Q(X_1, \dots, X_n) = 0 \}$$

in $\mathbb{C}\langle X_1, \dots, X_n \rangle$ and $\mathbb{C}\langle X_1, \dots, X_n \rangle^{\otimes 2}$, respectively. Then, for $j = 1, \dots, n$,

$$P + I_X^1 \longmapsto \partial_j P + I_X^2$$

induces a well-defined derivation

$$\hat{\partial}_j : \mathbb{C}\langle X_1,\dots,X_n\rangle / I_X^1 \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2} / I_X^2 .$$

Indeed, we have

$$P \in I_X^1 \implies \partial_j P \in I_X^2 .$$

For seeing this, take $P_1, P_2 \in \mathbb{C}\langle X_1,\dots,X_n\rangle$ and check that

$$\partial_j(P_1 P P_2) = (\partial_j P_1) P P_2 + P_1 (\partial_j P) P_2 + P_1 P (\partial_j P_2).$$

Since $P(X) = 0$, we get

$$(\partial_j(P_1 P P_2))(X) = P_1(X)(\partial_j P)(X) P_2(X)$$

and thus by assumption

$$\tau\otimes\tau\big((\partial_j(P_1 P P_2))(X)\big) = \tau\big((P_1 P P_2)(X)\,\xi_j\big) = 0$$

$$\|$$

$$\tau\otimes\tau\big(P_1(X)(\partial_j P)(X) P_2(X)\big) = \langle \partial_j P(X), P_1(X)^* \otimes P_2(X)^* \rangle$$

Hence, by linearity

$$\langle \partial_j P(X), Q \rangle = 0 \qquad \forall Q \in \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2},$$

which implies $\partial_j P(X) = 0$, i.e. $\partial_j P \in I_X^2$.

(2) Basic linear algebra shows that

$$\mathbb{C}\langle X_1,\dots,X_n\rangle / I_X^1 \xrightarrow[ev_X]{\sim} \mathbb{C}\langle X_1,\dots,X_n\rangle$$

and similarly

$$\mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}/I_X^2 \xrightarrow[ev_X\otimes ev_X]{\sim} \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}.$$

Thus, ① gives in fact a derivation

$$\hat{\partial}_j : \mathbb{C}\langle X_1,\dots,X_n\rangle \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2},$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}\langle X_1,\dots,X_n\rangle \arrow[r, "\partial_j"] \arrow[d, "ev_X"'] & \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2} \arrow[d, "ev_X\otimes ev_X"] \\
\mathbb{C}\langle X_1,\dots,X_n\rangle \arrow[r, "\hat{\partial}_j"] & \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}
\end{tikzcd}$$

In particular, we have for $i=1,\dots,n$

$$\begin{aligned}\hat{\partial}_j X_i &= \hat{\partial}_j(ev_X(X_i)) \\ &= ev_X\otimes ev_X(\partial_j X_i) = \delta_{i,j}\, 1\otimes 1.\end{aligned}$$

This shows (b).

③ Due to ①, we have for each $P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$

$$P(X)=0 \implies \forall j=1,\dots,n: \ \partial_j P(X)=0.$$

We define, for $j=1,\dots,n$,

$$\Delta_j : \mathbb{C}\langle X_1,\dots,X_n\rangle \longrightarrow \mathbb{C}\langle X_1,\dots,X_n\rangle$$

by $\Delta_j P := (\tau \otimes id)\big((ev_X \otimes id)(\partial_j P)\big)$. Thus

$$P(X) = 0 \implies \forall j = 1, \dots, n : (\Delta_j P)(X) = 0.$$

Now, take any $P \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ such that $P(X) = 0$ holds and write

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1, \dots, i_k} X_{i_1} \cdots X_{i_k}.$$

We observe that

$$\Delta_{i_d} \cdots \Delta_{i_1} P = a_{i_1, \dots, i_d} \cdot 1,$$

which gives

$$a_{i_1, \dots, i_d} \cdot 1 = (\Delta_{i_d} \cdots \Delta_{i_1} P)(X) = 0.$$

Hence, P is a constant polynomial, so we must have $P = 0$. This shows (a).

$\square$

<u>Question:</u>

If a conjugate system $(\xi_1, \dots, \xi_n)$ exists, can we also exclude <u>zero divisors</u>, i.e.

$$\left.\begin{array}{ll} P \in \mathbb{C}\langle X_1, \dots, X_n\rangle, & P \neq 0 \\ w \in vN(X_1, \dots, X_n), & w \neq 0 \end{array}\right\} \implies Pw \neq 0 \;?$$

YES we can!

6.3. Theorem:

Let (M,τ) be a tracial W^*-probability space and let $X_j = X_j^* \in M$, $j=1,\dots,n$, be given. Assume that a conjugate system $(\xi_1,\dots,\xi_n)$ for $X=(X_1,\dots,X_n)$ exists. Then, for any $0 \neq w \in vN(X_1,\dots,X_n)$ and each $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$, we have that

$$P(X_1,\dots,X_n)\,w = 0 \implies P = 0.$$

6.4. Corollary:

Under the assumptions of Theorem 6.3, the distribution measure $\mu_{P(X_1,\dots,X_n)}$ of $P(X_1,\dots,X_n)$ for any non-constant polynomial P does not have atoms.

Proof:

Put $Y := P(X_1,\dots,X_n)$. Recall that

$$\tau(Y^k) = \int_{\mathbb{R}} t^k \, d\mu_Y(t), \quad k = 0,1,2,\dots$$

determines μ_Y. If E_Y denotes the spectral measure of Y, i.e.

$$Y^k = \int_{\mathbb{R}} t^k \, dE_Y(t), \quad k = 0,1,2,\dots,$$

then $\mu_Y = \tau \circ E_Y$. Thus, if there is $\alpha \in \mathbb{R}$ such that $\mu_Y(\{\alpha\}) \neq 0$, then $w := E_Y(\{\alpha\}) \in vN(X_1, \dots, X_n)$ gives a non-zero projection with

$$(P(X_1, \dots, X_n) - \alpha 1)\, w = 0.$$

By Theorem 6.3, it follows $P - \alpha 1 = 0$, which contradicts the assumption that P is non-constant.

□

The proof of Theorem 6.3 needs some preparation.

<u>6.5. Lemma:</u>

In the situation of Theorem 6.3, we have for all $P_1, P_2 \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ and $j = 1, \dots, n$

(a) $\| \partial_j^* (P_1 \otimes P_2) \|_2 \leq 3 \| \xi_j \|_2 \, \| P_1 \| \, \| P_2 \|$

(b) $\| (id \otimes \tau)((\partial_j P_1) \cdot P_2) \|_2 \leq 4 \| \xi_j \|_2 \, \| P_1 \| \, \| P_2 \|$

and

$\| (\tau \otimes id)(P_1 \cdot (\partial_j P_2)) \|_2 \leq 4 \| \xi_j \|_2 \, \| P_1 \| \, \| P_2 \|$

<u>Proof:</u>

Use Dabrowski's inequalities, Theorem 3.19.

□

6.6. Lemma:

For any $w \in vN(X_1, \dots, X_n)$, there exists a sequence $(w_k)_{k=1}^{\infty}$ of elements in $\mathbb{C}\langle X_1, \dots, X_n \rangle$, such that

(i) $\sup_{k \in \mathbb{N}} \| w_k \| \leq \| w \|$.

(ii) $\| w_k - w \|_2 \longrightarrow 0$ for $k \to \infty$.

If $w = w^*$, we may assume that $w_k = w_k^*$.

Proof:

Use Kaplansky's density theorem. $\square$

6.7. Proposition:

Under the assumptions of Theorem 6.3, we have for all $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, $u, v \in vN(X_1, \dots, X_n)$

$$|\langle v^*(\partial_j P) u, Q_1 \otimes Q_2 \rangle|$$
$$\leq 4 \| \xi_j \|_2 \left(\| P u \|_2 \| v \| + \| u \| \| P^* v \|_2 \right) \| Q_1 \| \| Q_2 \|$$

for all $Q_1, Q_2 \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ and $j = 1, \dots, n$.

Proof:

First, we assume $u, v \in \mathbb{C}\langle X_1, \dots, X_n \rangle$ and we observe that

$$< Pu, \partial_j^*(v Q_1 \otimes Q_2) >$$
$$= < \partial_j(Pu), v Q_1 \otimes Q_2 >$$
$$= < (\partial_j P)u, v Q_1 \otimes Q_2 > + < P(\partial_j u), v Q_1 \otimes Q_2 >$$
$$= < v^*(\partial_j P)u, Q_1 \otimes Q_2 > + \underbrace{< (\partial_j u)Q_2^*, P^* v Q_1 \otimes 1 >}_{= < (id \otimes \tau)((\partial_j u)Q_2^*), P^* v Q_1 >}$$

Rearranging the terms yields by Lemma 6.5

$$|< v^*(\partial_j P)u, Q_1 \otimes Q_2 >|$$
$$\leq |< Pu, \partial_j^*(v Q_1 \otimes Q_2) >| + |< (id \otimes \tau)((\partial_j u)Q_2^*), P^* v Q_1 >|$$
$$\leq \|Pu\|_2 \underbrace{\|\partial_j^*(v Q_1 \otimes Q_2)\|_2}_{\substack{\leq 3 \|\xi_j\|_2 \|v Q_1\| \|Q_2\| \\ \leq 4 \|\xi_j\|_2 \|v\| \|Q_1\| \|Q_2\|}} + \underbrace{\|(id \otimes \tau)((\partial_j u) Q_2)\|_2}_{\leq 4 \|\xi_j\|_2 \|u\| \|Q_2\|} \underbrace{\|P^* v Q_1\|_2}_{\leq \|P^* v\|_2 \|Q_1\|}$$
$$\leq 4 \|\xi_j\|_2 \left(\|Pu\|_2 \|v\| + \|u\| \|P^* v\|_2 \right) \|Q_1\| \|Q_2\|.$$

Due to Lemma 6.6., the obtained inequality extends to arbitrary $u, v \in \mathcal{U}N(X_1, \dots, X_n)$.

$\square$

6.8. Corollary:

In the situation of Theorem 6.3, we have for $u, v \in \mathcal{U}N(X_1, \dots, X_n)$ and $P \in \mathbb{C}\langle X_1, \dots, X_n \rangle$

$$\left. \begin{aligned} P(X_1, \dots, X_n) u &= 0 \\ P(X_1, \dots, X_n)^* v &= 0 \end{aligned} \right\} \Longrightarrow \forall j = 1, \dots, n: \; v^* \partial_j P(X_1, \dots, X_n) u = 0$$

Proof:

Proposition 6.7 tells us that

$$< v^* \partial_j P(X_1, \ldots, X_n) u \, , \, Q_1 \otimes Q_2 > \; = 0$$

for all $Q_1, Q_2 \in \mathbb{C}<X_1, \ldots, X_n>$. Thus,

$$v^* \partial_j P(X_1, \ldots, X_n) u = 0. \qquad \square$$

6.9. Lemma:

Let (M, τ) be a tracial W^*-probability space and assume that $M \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For any $X \in M$, the projections $P_{\ker(X)}$ and $P_{\ker(X^*)}$ belong to M and we have

$$\tau(P_{\ker(X)}) = \tau(P_{\ker(X^*)}).$$

In particular: $\ker(X) \neq \{0\} \iff \ker(X^*) \neq \{0\}$

Proof:

Consider the polar decomposition of X,

$$X = V(X^*X)^{1/2} = (XX^*)^{1/2} V,$$

where V is a partial isometry (i.e. $V = VV^*V$) such that

$$V^*V = P_{\overline{\operatorname{ran}(X^*)}} \quad \text{and} \quad VV^* = P_{\overline{\operatorname{ran}(X)}}.$$

Hence

$$1 - V^*V = P_{\overline{\mathrm{ran}(X^*)}^\perp} = P_{\ker(X)} \quad \text{and}$$

$$1 - VV^* = P_{\overline{\mathrm{ran}(X)}^\perp} = P_{\ker(X^*)}$$

and thus $\tau(P_{\ker(X)}) = \tau(P_{\ker(X^*)})$. □

6.10. Corollary:

In the situation of Theorem 6.3, we have:

$$P(X)\underset{\neq 0}{w} = 0 \implies \exists \overset{\neq 0}{p} \in vN(X_1, \dots, X_n) \text{ projection}: (\Delta_{p,j} P)(X) w = 0, \; j = 1, \dots, n$$

for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, where we put

$$\Delta_{p,j} P := (\tau \otimes id)\big(p \otimes 1 \cdot (ev_X \otimes id)(\partial_j P)\big).$$

Proof:

Since $\{0\} \neq \mathrm{ran}(w) \subseteq \ker(P(X))$, we have

$$p := P_{\ker(P(X)^*)} \in vN(X_1, \dots, X_n)$$

and $p \neq 0$ by Lemma 6.9. Corollary 6.8 yields

$$\Delta_{p,j} P(X) w = (\tau \otimes id)(p \otimes 1 \cdot \partial_j P(X) \cdot 1 \otimes w) = 0.$$

□

Proof of Theorem 6.3:

We write again

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1, \dots, i_k} X_{i_1} \cdots X_{i_k}$$

By Corollary 6.10, we can find projections

$p_1, \dots, p_d \in vN(X_1, \dots, X_n) \setminus \{0\}$ such that

$$\underbrace{\Delta_{p_d, i_d} \cdots \Delta_{p_1, i_1} P(X)}\, w = 0$$

$$= a_{i_1, \dots, i_d}\, \tau(p_1) \cdots \tau(p_d)\, 1.$$

Hence $a_{i_1, \dots, i_d} = 0$, which implies $P = 0$. $\square$