Functional Analysis II

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Contents

0 A brief reminder on Hilbert spaces and operators on Hilbert spaces

Throughout these lecture notes, the analytic closure in a topological $(X, \mathfrak{T})$ for $A \subseteq X$ will be denoted $\mathrm{cl}_{\mathfrak{T}}(A)$ or just $\mathrm{cl}(A)$, if no confusion is to be feared.

Definition 0.1: A *pre-Hilbert space* $(H, \langle \cdot, \cdot \rangle)$ is a $\mathbb{K}$-vector space H (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) that is endowed with an inner product $\langle \cdot, \cdot \rangle$, i. e. a map

$$\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathbb{K}$$

that satisfies

(i) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$ for all $x, y, z \in H$, $\lambda, \mu \in \mathbb{K}$,

(ii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in H$,

(iii) $\langle x, x \rangle \geq 0$ for all $x \in H$,

(iv) If $\langle x, x \rangle = 0$, then $x = 0$.

A *Hilbert space* $(H, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space $(H, \langle \cdot, \cdot \rangle)$ that is complete (i. e. a Banach space) with respect to the norm

$$\begin{aligned} \|\cdot\| : H &\longrightarrow [0, \infty) \\ x &\longmapsto \langle x, x \rangle^{\frac{1}{2}} \end{aligned}$$

that is induced by the inner product.

Remark 0.2 (Properties of pre-Hilbert spaces): Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space over $\mathbb{K}$.

(i) For all $x, y, z \in H$, $\lambda, \mu \in \mathbb{K}$, we have that

$$\langle z, \lambda x + \mu y \rangle = \overline{\lambda} \langle z, x \rangle + \overline{\mu} \langle z, y \rangle.$$

(ii) $\|\cdot\|$ is indeed a norm and the *Cauchy-Schwarz inequality* holds, i. e. for all $x, y \in H$:

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

(iii) $\|\cdot\|$ satisfies the *parallelogram identity*, i. e. for all $x, y \in H$:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

In fact, a normed space is a pre-Hilbert space if and only if the norm satsifies the parallelogram identity. Indeed, the inner product can be recovered by the *polarisation identities*

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \|x + \mathrm{i}^k y\|^2 \qquad \text{(if } \mathbb{K} = \mathbb{C}\text{)},$$

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \qquad\qquad (\text{if } \mathbb{K} = \mathbb{R}).$$

(iv) If $x, y \in H$ are *orthogonal* ($x \perp y$), i. e. $\langle x, y\rangle = 0$, then the *Pythagorean identiy* holds, i. e.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

(v) The completion of a pre-Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

Remark 0.3 (Properties of Hilbert spaces): Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space.

(i) If $C \subseteq H$ is closed and convex, then for each $x \in H \setminus C$ there is a unique point $y_0 \in C$, called the *best approximation to x* such that

$$\|x - y_0\| = \operatorname{dist}(x, C) := \inf_{y\in C} \|x - y\|.$$

(ii) If $K \subseteq H$ is a closed linear subspace, then $y_0 \in K$ is the best approximation to $x \in H \setminus K$ if and only if $\langle x - y_0, y\rangle = 0$ for all $y \in K$.

(iii) Let $M \subseteq H$ be any subset. We call

$$M^\perp := \{x \in H \mid \forall\, y \in M : \langle x, y\rangle = 0\}$$

the *orthogonal complement of M in H*. Note that $M^\perp$ is a closed linear subspace of H.

(iv) Let $K_1, K_2 \subseteq H$ be closed linear subspaces with $K_1 \perp K_2$, i. e. $\langle y_1, y_2\rangle = 0$ for all $y_1 \in K_1, y_2 \in K_2$. Then the *(orthogonal) direct sum*

$$K_1 \oplus K_2 := \{y_1 + y_2 \mid y_1 \in K_1, y_2 \in K_2\}$$

is again a closed linear subspace of H. Furthermore, $K_1 \cap K_2 = \{0\}$ and each $y \in K_1 \oplus K_2$ has a unique decomposition $y = y_1 + y_2$ with $y_1 \in K_1$ and $y_2 \in K_2$.

(v) The *projection theorem* (Theorem 5.18, Functional Analysis I) says that H decomposes as $H = K \oplus K^\perp$ for any closed linear subspace $K \subseteq H$; in fact each $x \in H$ can be written as $x = x_1 + x_2$, where $x_1 \in K$ and $x_2 \in K^\perp$ are the best approximations in K and $K^\perp$ respectively to x.

(vi) The *Riesz representation theorem* (Theorem 5.20, Functional Analysis I) says that the dual space $H' := B(H, \mathbb{K}) = \{f : H \to \mathbb{K} \text{ linear and bounded}\}$ can be identified with H via the anti-linear isometric isomorphism

$$\begin{aligned} j : H &\longrightarrow H' \\ y &\longmapsto f_y := \langle\cdot, y\rangle. \end{aligned}$$

In particular $H \cong H''$, i. e. H is reflexive.

(vii) A family $(e_i)_{i\in I}$ is called *orthonormal system*, if $\langle e_i, e_j\rangle = \delta_{i,j}$ for all $i, j \in I$. For such $(e_i)_{i\in I}$, *Bessels inequality* holds, i. e. for all $x \in H$ it holds

$$\sum_{i\in I} |\langle x, e_i\rangle|^2 \leq \|x\|^2$$

with equality if and only if $x = \sum_{i\in I}\langle x, e_i\rangle e_i$ (Theorem 5.27, Functional Analysis I). We call $(e_i)_{i\in I}$ an *orthonormal basis*, if one of the following equivalent conditions is satisfied:[1]

(1) $(e_i)_{i\in I}$ is a maximal orthonormal system,
(2) If $x \perp e_i$ for all $i \in I$, then $x = 0$,
(3) For all $x \in H$ it holds $x = \sum_{i\in I}\langle x, e_i\rangle e_i$,
(4) For all $x \in H$ it holds $\|x\|^2 = \sum_{i\in I}|\langle x, e_i\rangle|^2$,
(5) The set $\{\sum_{i\in F} \alpha_i e_i \mid F \subseteq_{\text{fin}} I, \alpha_i \in \mathbb{K}\}$ is dense in H.

If $(f_j)_{j\in J}$ is another orthonormal basis of H, then $|I| = |J|$; the unique cardinality of an orthonormal basis of H is called the *(Hilbert space) dimension of* H, denoted by $\dim H$. Every Hilbert space admits an orthonormal basis. Two Hilbert spaces H and K are isomorphic ($H \cong K$), i. e. there is a sujective linear map $U : H \to K$ that satisfies

$$\langle Ux, Uy\rangle_K = \langle x, y\rangle_H \qquad \text{for all } x, y \in H,$$

if and only if $\dim H = \dim K$.

If $\dim H$ is countable, we call H *separable*; this is equivalent to H being separable as a Banach space (i. e. there is a countable dense subset).

Example 0.4: (i) $\mathbb{K}^n$ is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_j \overline{y}_j \qquad \text{for } x = (x_1, \dots, x_n)^\top, y \in (y_1, \dots, y_n)^\top \in \mathbb{K}^n.$$

(ii) If $(\Omega, \mathfrak{F}, \mu)$ is a measure space, then

$$L^2(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{C} \text{ measurable} : \int_\Omega |f(\omega)|^2 \, d\mu(\omega) < \infty \right\} / \mathcal{N},$$

where $\mathcal{N} := \{f : \Omega \to \mathbb{C} \text{ measurable} \mid \mu(\{\omega \in \Omega : f(\omega) \neq 0\}) = 0\}$, is a Hilbert space with inner product

$$\langle f, g\rangle = \int_\Omega f(\omega)\overline{g(\omega)} \, d\mu \qquad \text{for all } f, g \in L^2(\Omega, \mu).$$

[1] This is the so called theorem of Parseval, refer to Theorem 5.28 from the Functional Analysis I lecture notes.

In particular, if a set I is endowed with the *counting measure* σ_I, we obtain the Hilbert space $\ell^2(I) := L^2(I, \sigma_I)$ with $\dim \ell^2(I) = |I|$. Each separable Hilbert space H that is not finite dimensional satisfies $H \cong \ell^2(\mathbb{N})$.

An orthonormal basis of $L^2([0, 2\pi), \lambda^1)$ with the *Lebesgue measure* λ^1 on $[0, 2\pi)$ is given by $(e_n)_{n\in\mathbb{Z}}$ where

$$e_n(t) = \frac{1}{\sqrt{2\pi}} e^{\mathrm{i}nt} \qquad\qquad \text{for } t \in [0, 2\pi).$$

For $f \in L^2([0, 2\pi), \lambda^1)$, the representation $f = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n$ (in the L^2-sense) is called the *Fourier series* of f and

$$\hat{f}_n := \frac{1}{\sqrt{2\pi}} \langle f, e_n\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-\mathrm{i}nt} \, d\lambda^1(t)$$

are called the *Fourier coefficients*; in particular $\|f\|^2 = 2\pi \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2$.

(iii) *Hardy spaces* and *Bergman spaces* are Hilbert spaces of holomorphic functions, especially so-called *reproducing Hilbert spaces.*

Remark 0.5: Let X be a vector space over $\mathbb{K}$.

(i) A map $p : X \to [0, \infty)$ is called a *seminorm*, if it satisfies the following statements:

- For all $x \in X$ and $\lambda \in \mathbb{K}$ it holds $p(\lambda x) = |\lambda| p(x)$,
- For all $x, y \in X$ it holds $p(x + y) \leq p(x) + p(y)$

For $x \in X$ and $r > 0$, we put $B_p(x, r) := \{y \in X \mid p(y - x) < r\}$.

(ii) We call X

- *topological vector space*, if X is endowed with a topology $\mathfrak{T}$ with respect to which

$$\begin{aligned} + : X \times X &\longrightarrow X & \qquad \cdot : \mathbb{K} \times X &\longrightarrow X \\ (x, y) &\longmapsto x + y & (\alpha, x) &\longmapsto \alpha x \end{aligned}$$

  are both continuous.
- *locally convex vector space*, if X is a topological vector space whose topology $\mathfrak{T}$ is generated by some family P of seminorms, i. e. $U \subseteq X$ is open in $(X, \mathfrak{T})$ if and only if

$$\forall\, x \in U \,\exists\, n \in \mathbb{N} \,\exists\, p_1, \ldots, p_n \in P \,\exists\, \varepsilon_1, \ldots, \varepsilon_n > 0 : \bigcap_{i=1}^{n} B_{p_i}(x, \varepsilon_i) \subseteq U.$$

That topology is Hausdorff if and only if for all $0 \neq x \in X$, there is $p \in P$ such that $p(x) > 0$.

Definition 0.6: Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$.

(i) The topology induced by $\|\cdot\|$ is called the *strong topology on H*.

(ii) The locally convex (Hausdorff) topology induced by the family $P = \{p_x \mid x \in H\}$ of seminorms

$$\begin{aligned} p_x : H &\longrightarrow [0, \infty) \\ y &\longmapsto |\langle y, x \rangle|, \end{aligned}$$

is called the *weak topology on H*.

Theorem 0.7: *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$.*

(i) *If a sequence $(x_n)_{n \in \mathbb{N}}$ in H converges weakly to some point $x \in H$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $(x_n)_{n \in \mathbb{N}}$ converges strongly to x.*

(ii) *Every bounded sequence in H has a weakly convergent subsequence.*

(iii) *Every bounded sequence $(x_n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that*

$$\Big(\frac{1}{K} \sum_{k=1}^{K} x_{n_k} \Big)_{K \in \mathbb{N}}$$

converges strongly (Theorem of Banach-Saks).

Proof: Part (ii) and (iii) are exercises, we want to show part (i). We have

$$\|x - x_n\|^2 = \|x\|^2 - 2\mathrm{Re}(\langle x_n, x \rangle) + \|x_n\|^2$$

and by weak convergence of $(x_n)_{n \in \mathbb{N}}$ we also have

$$|\langle x_n, x \rangle - \|x\|^2| = |\langle x_n - x, x \rangle| = p_x(x_n - x) \longrightarrow 0 \text{ as } n \to \infty,$$

i. e. $\langle x_n, x \rangle \to \|x\|^2$ as $n \to \infty$, hence $\mathrm{Re}(\langle x_n, x \rangle) \to \|x\|^2$ as $n \to \infty$. Thus it holds $\|x - x_n\| \to 0$ as $n \to \infty$. ■

Remark 0.8: (i) One can show that the closed unit ball $\{x \in H \mid \|x\| \leq 1\}$ is compact with respect to the weak topology.

(ii) If the Hilbert space is separable, then the weak topology on $\{x \in H \mid \|x\| \leq 1\}$ is metrisable.

Remark 0.9: (i) Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(K, \langle \cdot, \cdot \rangle_K)$ be Hilbert spaces over $\mathbb{K}$. For each $A \in B(H, K)$, there is a unique operator $A^* \in B(K, H)$ with

$$\langle Ax, y \rangle_K = \langle x, A^* y \rangle_H \qquad \text{for all } x \in H, y \in K.$$

We call A^* the *adjoint operator to A*; with $A' \in B(K', H')$ defined via $A' f := f \circ A$, it is given as $A^* = j_H^{-1} \circ A' \circ j_K$ and we have that $\|A^*\| = \|A\|$.[2]

[2] For the notation, refer to the proof of Proposition 7.4 in the lecture notes of Functional Analysis I.

(ii) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. Then $B(H) := B(H,H)$ is a Banach algebra with respect to the operator norm $\|\cdot\|$, in particular $\|ST\| \leq \|S\|\|T\|$ for all $S, T \in B(H)$. The map $^* : B(H) \to B(H)$, $A \mapsto A^*$ satisfies

- $(\lambda A + \mu B)^* = \overline{\lambda} A^* + \overline{\mu} B^*$,
- $(AB)^* = B^* A^*$,
- $(A^*)^* = A$,
- $\|A^*\| = \|A\|$ and $\|A^* A\| = \|A\|^2$.

(iii) Note that $\ker(A) = \operatorname{im}(A^*)^\perp$ for all $A \in B(H)$.

Definition 0.10: Let $(H, \langle \cdot, \cdot \rangle_H)$ and $(K, \langle \cdot, \cdot \rangle_K)$ be Hilbert spaces.

(i) $A \in B(H)$ is called *selfadjoint (or hermitian)*, if $A = A^*$.

(ii) $A \in B(H)$ is called *normal*, if $AA^* = A^* A$.

(iii) $A \in B(H)$ is called *positive*, if $A = A^*$ and $\langle Ax, x \rangle \geq 0$ for all $x \in H$.

(iv) $V \in B(H,K)$ is called *isometry* if one of the following equivalent statements hold:

- $V^* V = \mathrm{id}_H$,
- $\|Vx\|_K = \|x\|_H$ for all $x \in H$,
- $\langle Vx, Vy \rangle_K = \langle x, y \rangle_H$ for all $x, y \in H$.

(v) $U \in B(H,K)$ is called *unitary* if $U^* U = \mathrm{id}_H$ and $UU^* = \mathrm{id}_K$ or equivalently if U is a surjective isometry.

(vi) $P \in B(H)$ is called *(orthogonal) projection* if $P^2 = P = P^*$. Then $\operatorname{im}(P)$ is a closed linear subspace of H (in fact $\operatorname{im}(P) = \ker(1 - P)$ where $1 = \mathrm{id}_H$) and $H = \operatorname{im}(P) \oplus \ker(P)$.

Conversely: If $K \subseteq H$ is a closed linear subspace, then

$$\begin{aligned} P : H = K \oplus K^\perp &\longrightarrow H \\ x = x_1 + x_2 &\longmapsto x_1 \end{aligned}$$

is an orthogonal projection; note that Px is the best approximation to x in K.

(vii) $V \in B(H)$ is called *partial isometry* if one of the following equivalent conditions holds:

- $VV^* V = V$,
- $V^* V$ is a projection (*initial projection*),
- VV^* is a projection (*final projection*),
- There is a closed linear subspace $K \subseteq H$ such that $V|_K : K \to H$ is an isometry and $V|_{K^\perp} \equiv 0$.[3]

[3] Check Functional Analysis I, Sheet 10, Exercise 3.

Theorem 0.11 (Polar decomposition): *For each $T \in B(H)$, there is a unique partial isometry $V \in B(H)$ such that $T = V|T|$ and $\ker(V) = \ker(T)$ where we define $|T| := (T^*T)^{\frac{1}{2}}$.*

In fact, V^*V is the projection onto $\ker(T)^\perp$ and VV^* is the projection onto $\mathrm{cl}(\mathrm{im}(T))$[4]

[4]Check Functional Analysis I, Sheet 11, Exercise 3.

1 Locally convex topologies on the space of operators

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$.

Definition 1.1: (i) The *weak operator topology* (WOT) on $B(H)$ is the locally convex topology that is defined by the family of seminorms

$$\begin{aligned} p_{x,y} : B(H) &\longrightarrow [0,\infty) \\ T &\longmapsto |\langle Tx, y\rangle| \qquad \text{for all } x, y \in H. \end{aligned}$$

(ii) The *strong operator topology* (SOT) on $B(H)$ is the locally convex topology induced by the family of seminorms

$$\begin{aligned} p_{x} : B(H) &\longrightarrow [0,\infty) \\ T &\longmapsto \|Tx\| \qquad \text{for all } x, y \in H. \end{aligned}$$

(iii) The *operator norm topology* (ONT) on $B(H)$ is the topology induced by the operator norm $\|\cdot\|$.

Remark 1.2: We have $\mathfrak{T}_{\mathrm{WOT}} \subseteq \mathfrak{T}_{\mathrm{SOT}} \subseteq \mathfrak{T}_{\mathrm{ONT}}$; note that

$$|\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\| \qquad \forall\, x, y \in H, T \in B(H).$$

Remark 1.3: (i) $T \mapsto T^*$ is continuous with respect to ONT and WOT, but is not continuous with respect to SOT.

(ii) Multiplication $\cdot : B(H) \times B(H) \to B(H)$, $(S, T) \mapsto ST$ is continuous with respect to ONT but not continuous with respect to SOT or WOT.

(iii) However, for fixed $S \in B(H)$, both mappings $T \mapsto ST$ and $T \mapsto TS$ are continuous with respect to SOT and WOT.

(iv) Furthermore: If $(S_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$ are sequences in $B(H)$ that are strongly convergent to S and T respectively, then $(S_nT_n)_{n\in\mathbb{N}}$ converges strongly to ST. Indeed, $(\|S_n\|)_{n\in\mathbb{N}}$, $(\|T_n\|)_{n\in\mathbb{N}}$ are bounded by the uniform boundedness principle, so that

$$\|S_nT_nx - STx\| \leq \|S_n\|\|(T_n - T)x\| + \|(S_n - S)x\| \longrightarrow 0 \text{ as } n \to \infty.$$

If now the strong— is replaced by the weak topology, the statement is not valid anymore.

Operator algebras: ONT $\rightsquigarrow$ C^*-algebras, WOT, SOT $\rightsquigarrow$ von Neumann algebras.

2 Unitisation of C^*-algebras

Definition 2.1: (i) A C^**-algebra* is a Banach algebra A (i. e. A is a $\mathbb{C}$-algebra that is complete and normed such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$) with an involution, i. e. an anti-linear map

$$\begin{aligned} ^*\colon A &\longrightarrow A \\ x &\longmapsto x^* \end{aligned}$$

such that $x^{**} = x$, $(xy)^* = y^*x^*$ and $\|x^*x\| = \|x\|^2$.

(ii) A is *unital*, if A is unital as an algebra ($1 \in A$).

Motivation 2.2: (i) So far, we only considered unital C^*-algebras such as

$$B(H) := \{T\colon H \longrightarrow H \text{ linear, bounded}\},$$

for instance $M_n(\mathbb{C}) = B(\mathbb{C}^n)$ or $C(X) := \{f\colon X \to \mathbb{C} \text{ continuous}\}$ for a compact metric space X. These are natural as well as unital examples and we have strong theorems for them:

Theorem (First funamental theorem for C^*-algebras): *Let A be a commutative and unital C^*-algebra. Then $A \cong C(X)$ for some compact metric space X.*

In the proof of the Stone-Weierstraß Theorem, we also needed the unit:

$$\sqrt{f} = \sqrt{1-g} = 1 - \sum_{n=1}^{\infty} a_n g^n \leftarrow 1 - \sum_{n=1}^{N} a_n g^n \in A$$

for $0 \leq f \leq 1$, $f \in A$ and $g := 1 - f$.

(ii) The compact operators $K(H) := \{T\colon H \to H \text{ compact}\} \subseteq B(H)$ satisfy Definition 2.1 (i), but *not* (ii). One checks, that $K(H)$ is infact a C^*-algebra.
More general: If A is a C^*-algebra and $I \triangleleft A$ an ideal (two-sided and closed, $I^* \subseteq I$), then I is a (non-unital) C^*-algebra. Hence, there are important examples of non-unital C^*-algebras.

(iii) Another important example (for a unital C^*-algebra) was $C(X)$ for a compact metric space X. Why "compact"?

$$\|f\|_\infty := \sup\{|f(t)| \mid t \in X\} < \infty \text{ if } X \text{ is compact.}$$

Let now X be locally compact (i. e. for all $x \in X$ and $U \subseteq X$ open with $x \in U$, there is a compact neighbourhood $K \subseteq X$ such that $x \in K \subseteq U$. Then $\|f\|_\infty = \infty$ for $f \in C(X)$ is possible.

Consider

$$\begin{aligned} &C_0(X)\\ &:= \{f\colon X \to \mathbb{C} \text{ continuous, vanishing at } \infty\}\\ &\;= \{f\colon X \to \mathbb{C} \text{ continuous, } \forall \varepsilon > 0 \,\exists K \subseteq X \text{compact} : |f(t)| < \varepsilon \,\forall t \in X \setminus K\}, \end{aligned}$$

then $\|f\|_\infty < \infty$ for all $f \in C_0(X)$. One can check that $(C_0(X), \|\cdot\|_\infty)$ is a commutative (non-unital) C^*-algebra.

(iv) We shall find ways of dealing with non-unital C^*-algebras, for instance by "adding a unit": a minimal unit, a maximal unit or an approximitive unit. Amongst other, we need to study how to find C^*-norms.

Proposition 2.3: *Let B be a unital C^*-algebra.*

(i) *Let $x \in B$ with $x = x^*$. Then $r(x) = \|x\|$.*[1] ("the norm is algebraic")

(ii) *The C^*-norm on B is unique.*

(iii) *If A is a unital *-Banach algebra such that $\|x^*\| = \|x\|$ and $\|1\| = 1$ and $\varphi\colon A \to B$ is a unital *-homomorphism (i. e. $\varphi : A \to B$ is an algebra-homomorphism that satisfies $\varphi(x)^* = \varphi(x^*)$), then $\|\varphi(x)\| \le \|x\|$.*

(iv) *On a unital *-Banach algebra, the C^*-norm (if it exists) is the minimal norm.*

Proof: (i) From Theorem 8.13, Functional Analysis I, we know that

$$r(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|} = \lim_{n\to\infty} \sqrt[2^n]{\|x^{2^n}\|} = \lim_{n\to\infty} \sqrt[2^n]{\|x\|^{2^n}} = \|x\|,$$

as $\|x^{2^{n+1}}\| = \|(x^{2^n})^*(x^{2^n})\| = \|x^{2^n}\|^2 = \|x\|^{2^{n+1}}$ via induction.

(ii) Let $\|\cdot\|_1$, $\|\cdot\|_2$ be C^*-norms on B. Then

$$\|x\|_1^2 = \|x^*x\|_1 \overset{\text{(i)}}{=} r(x^*x) = \|x^*x\|_2 = \|x\|_2^2.$$

(iii) We can estimate

$$\|\varphi(x)\|_B^2 = \|\varphi(x^*x)\|_B \overset{\text{(i)}}{=} r(\varphi(x^*x)) \le r(x^*x) \le \|x^*x\|_A \le \|x^*\|_A\|x\|_A = \|x\|_A^2$$

because if $\lambda - \varphi(y) = \varphi(\lambda 1 - y)$ is not invertible, then $\lambda 1 - y$ is not invertible: If z is invertible, then there is z^{-1} such that $zz^{-1} = 1$, thus $\varphi(z)\varphi(z^{-1}) = 1$.

(iv) The map

$$\begin{aligned} \varphi\colon (A, \|\cdot\|_{\text{Banach norm}}) &\longrightarrow (A, \|\cdot\|_{C^*-\text{algebra}})\\ x &\longmapsto x \end{aligned}$$

is a *-homomorphism, hence $\|x\|_{C^*-\text{algebra}} = \|\varphi(x)\| \le \|x\|_{\text{Banach norm}}$. ■

[1] Note that the spectral radius is defined as $r(x) := \max\{|\lambda| \mid \lambda \in \mathrm{Sp}(x)\} \le \|x\|$ and the spectrum $\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} \mid \lambda 1 - x \text{ not invertible in } B\} \subseteq \mathbb{C}$ is compact. Refer to Proposition 8.8 from Functional Analysis I.

Reminder 2.4 (Theorem of Gelfand-Naimark[2]): Let A be a commutative unital C^*-algebra. Then the Gelfand transformation

$$\begin{aligned} \chi\colon A &\longrightarrow C(\operatorname{Spec}(A)) && (\operatorname{Spec}(A) := \{\varphi\colon A \to \mathbb{C} \text{ unital algebra homomorphism}\}) \\ x &\longmapsto \hat{x} && (\hat{x}(\varphi) := \varphi(x)) \end{aligned}$$

is an isometric *-isomorphism.

Spec(A) is compact, refer to Proposition 10.13 of Functional Analysis I.

Proof: χ is a *-algebra homomorphism because φ is an algebra homomorphism; Lemma 10.10 shows that χ respects *. χ is isometric (and thus injective), because

$$\|\hat{x}\|_\infty^2 = \|\chi(x^*x)\|_\infty = r(x^*x) = \|x^*x\| = \|x\|^2$$

where you use Corollary 10.12 from Functional Analysis I for the second equality. χ is surjective because of the Stone-Weierstraß Theorem. ■

Remark 2.5: (i) Gelfand-Naimarks's Theorem (from the 1940's) is a milestone in the theory of C^*-algebras since:

(1) It justifies the view point of "non-commutative topology".

(2) It yields the very useful tool of "functional calculus".

Can we get (1) and (2) in the non-unital situation, too?

(ii) About "non-commutative topology" as a part of the "non-commutative world":

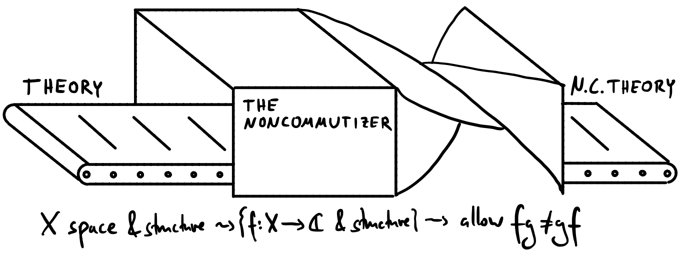


Figure 2.1: The noncommutiser.

$$\begin{aligned} \text{Topology} &\longrightarrow C^*\text{-algebras} \\ \text{Measure theory} &\longrightarrow \text{von Neumann algebras} \\ \text{Probability theory / Independce} &\longrightarrow \text{Free Probability / Free Independence} \end{aligned}$$

[2] We call it the first fundamental theorem of C^*-algebras. This is non-standard naming.

$$
\begin{array}{rcl}
\text{Differential Geometry} & \longrightarrow & \text{Noncommutative Geometry} \\
\text{Symmetry / Groups} & \longrightarrow & \text{Quantum symmetry / Quantum groups}
\end{array}
$$

– –

$$
\begin{array}{rcl}
\text{Information theory} & \longrightarrow & \text{Quantum information} \\
\text{Complex analysis} & \longrightarrow & \text{Free Analysis}
\end{array}
$$

(iii) A dictionary of non-commutative topolgy:

topology (X locally compact)	C^*-algebra ($C_0(X)$)	
compact	unital	
open subset	closed ideal	
closed subset	quotient	
metrisable	separable	
connected	projectionless	(no trivial projections)

See Gracia-Bondia, Vanilly, Figuenoa : Elements of non-commutative geometry, Introduction to chapter 1 or end of chapter 3; see also Wegge-Olsen, Chapter 1.11.

(iv) History / Importance of C^*-algebras:

- The name "C^*-algebras" was introduced by Segal in 1947 where "C" stands for "continuous" and "$*$" stands for "involution".
- The study of C^*-algebras may be seen as the study of operators on a Hilbert space by algebraic means (refer to the Introduction of "C^*-algebras and their automorphism groups" by Pedersen). Reminder: Suppose that $A \subseteq B(H)$ is a subalgebra such that "$x \in A \Rightarrow x^* \in A$" holds. Then $\mathrm{cl}_{\|\cdot\|}(A) \subseteq B(H)$ is a C^*-algebra and $\mathrm{cl}_{\mathrm{SOT}}(A)$ and $\mathrm{cl}_{\mathrm{WOT}}(A)$ are von Neumann algebras. One can show: There always is a faithful representation

$$\pi : A \hookrightarrow B(H).$$

Lemma 2.6: (i) *Let A and B be C^*-algebras. Then $A \oplus B$ is again a C^*-algebra via*

$$A \oplus B := \{(a, b) \mid a \in A, b \in B\}$$

with pointwise operations and $\|(x, y)\| := \max\{\|x\|, \|y\|\}$.

(ii) *Let A be a *-Banach algebra. Then*

$$\tilde{A} := \{(x, \lambda) \mid x \in A, \lambda \in \mathbb{C}\}$$

is a *-Banach algebra with unit $(0, 1)$ *with the operations*

$$
\begin{aligned}
(x, \lambda) + (y, \mu) &:= (x + y, \lambda + \mu), \\
(x, \lambda)^* &:= (x^*, \overline{\lambda})
\end{aligned}
$$

$$(x, \lambda)(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$$

and the norm

$$\|(x, \lambda)\|_{BA} := \|x\| + |\lambda|.$$

Proof: For the Banach property of the new algebra defined in (ii):

$$\|(x, \lambda)(y, \mu)\| \leq \|x\|\|y\| + |\lambda|\|y\| + |\mu|\|x\| + |\lambda||\mu| = \|(x, \lambda)\|\|(y, \mu)\|. \qquad \blacksquare$$

Note that $(\tilde{A}, \|\cdot\|_{BA})$ is a unital *-Banach algebra, but *not* a C^*-algebra, since $\|(x, \lambda)^*(x, \lambda)\| \neq \|(x, \lambda)\|^2$. We write $\lambda 1 + x := (x, \lambda)$ for the elements in $\tilde{A}$ making the multiplication intuitive. The embedding

$$\begin{aligned} A &\hookrightarrow \tilde{A} \\ x &\longmapsto (x, 0) \end{aligned}$$

is an injective *-homomorphism.

Theorem 2.7: *Let A be a C^*-algebra. On $\tilde{A}$ there is a unique norm turning it into a C^*-algebra. Then*

$$\begin{aligned} A &\hookrightarrow \tilde{A} \\ x &\longmapsto (x, 0) \end{aligned}$$

is isometric.

Proof: The uniqueness is granted by Proposition 2.3, we thus only have to show exsistence.

(1) Let A be unital with unit e. Then

$$\begin{aligned} \tilde{A} &\longrightarrow A \oplus \mathbb{C} \\ (x, \lambda) &\longmapsto (\lambda e + x, \lambda) \end{aligned}$$

is a bijective unital *-algebra homomorphism (so $\tilde{A} \cong A \oplus \mathbb{C}$ as *-algebras). Hence $\tilde{A}$ has a C^*-norm, namely the one of $A \oplus \mathbb{C}$.

(2) Let A be non-unital. For $x \in \tilde{A}$, write $x = \lambda 1 + a$. Consider

$$L : \tilde{A} \longrightarrow B(A) := \{T : A \to A \text{ bounded, linear}\}$$

where $L_x(b) := xb = \lambda b + ab$ for $b \in A$. The linearity of L is clear. For the boundedness we notice that

$$\|L_x(b)\| = \|xb\|_A = \|\lambda b + ab\|_A \leq (|\lambda| + \|a\|_A)\|b\|_A$$

and thus $\|L_x\| \leq |\lambda| + \|a\| = \|x\|_{BA}$. Now put

$$\|x\|_{\tilde{A}} := \|L_x\| = \sup\{\|\lambda 1 + a)b\|_A \mid b \in A, \|b\| \leq 1\}$$

with $x = \lambda 1 + a$. We have $\|a\|_{\tilde{A}} = \|a\|_A$ for all $a \in A$, as

$$\|a\|\|a^*\| = \|a\|^2 = \|aa^*\| = \|L_a(a^*)\| \leq \|L_a\|\|a^*\|$$

and thus $\|a\|_A \leq \|a\|_{\tilde{A}}$. On the other hand we know that $\|a\|_{\tilde{A}} = \|L_a\| \leq \|a\|_A$ from the proof that L_a is bounded.

To show that $\|\cdot\|_{\tilde{A}}$ is a norm, we only need to show that "$\|x\|_A = 0 \Rightarrow x = 0$" holds. Let $x \in \tilde{A}$ such that $x = \lambda 1 + a$, $\lambda \neq 0$. Assume that $\|L_x\| = 0$, hence $xb = 0$ for all $b \in A$ and thus $\lambda b + ab = 0$ for all $b \in A$. Therefore, $-\lambda^{-1}a$ is a unit in A, since $b = -\lambda^{-1}ab$. This is a contradiction as A was assumed to be non-unital.

Now we need to show that $\|\cdot\|_{\tilde{A}}$ is submultiplicative: It holds that

$$\|L_{xy}\| = \|L_x L_y\| \leq \|L_x\|\|L_y\|.$$

$(\tilde{A}, \|\cdot\|_{\tilde{A}})$ is in fact a *-Banach algebra, but we will omit the proof of completeness.

Finally, $\|\cdot\|_{\tilde{A}}$ satisfies the C^*-condition: Let $x \in \tilde{A}$ and $\varepsilon > 0$. There is a $b \in A$ with $\|b\| \leq 1$ such that $\|xb\|_A \geq \|L_x\| - \varepsilon$, thus

$$(\|L_x\| - \varepsilon)^2 \leq \|xb\|_A^2 = \|b^* x^* x b\|_A \leq \|b^*\|_A \|L_{x^*x}(b)\|_A \leq \|L_{x^*x}\|.$$

Therefore we get $\|x\|_{\tilde{A}}^2 = \|L_x\|^2 \leq \|L_{x^*x}\| = \|x^*x\|_{\tilde{A}}$. In general it holds: $\|x\|^2 \leq \|x^*x\| \Rightarrow \|x^*x\|$, because

$$\begin{aligned} \|x\|^2 \leq \|x^*x\| \leq \|x^*\|\|x\| \Rightarrow \|x\| \leq \|x^*\| \leq \|x^{**}\| = \|x\| \\ \Rightarrow \|x\|^2 \leq \|x^*x\| \leq \|x\|^2. \end{aligned}$$

■

Remark 2.8: This is not the only unitisation of a C^*-algebra:

(i) Another possibiliy comes from the *multiplier algebra*

$$\mathcal{M}(A) := \{(L, R) \text{ double centralisers}\}$$

with $L, R \in B(A)$ such that $L(ab) = L(a)b$, $R(ab) = aR(b)$ and $R(a)b = aL(b)$. For instance let $L_x(a) := xa$ and $R_x(a) := ax$, then $(L_x, R_x) \in \mathcal{M}(A)$. We have $\|L\| = \|R\|$. $\mathcal{M}(A)$ is a C^*-algebra via

$$\begin{aligned} (L_1, R_1) + (L_2, R_2) &:= (L_1 + L_2, R_1 + R_2) \\ \lambda(L, R) &:= (\lambda L, \lambda R) \\ (L_1, R_1)(L_2, R_2) &:= (L_1 L_2, R_2 R_1) \\ (L, R)^* &:= (R^*, L^*) \\ L^*(a) &:= L(a^*)^* \end{aligned}$$

$$
\begin{aligned}
R^*(a) &:= R(a^*)^* \\
\|(L, R)\| &:= \|L\| = \|R\| \\
1 &:= (\mathrm{id}, \mathrm{id}).
\end{aligned}
$$

Then $\mathcal{M}(A)$ is a unital C^*-algebra and

$$
\begin{aligned}
A &\hookrightarrow \mathcal{M}(A) \\
x &\longmapsto (L_x, R_x)
\end{aligned}
$$

is an isometric embedding.

(ii) In fact $A \subseteq \tilde{A}$ and $A \subseteq \mathcal{M}(A)$ are ideals.

(iii) Let A, B be C^*-algebras, B unital, $A \subseteq B$ as an ideal. Then the diagrams

$$
\begin{tikzcd}
A \arrow[rr, "\subseteq"] \arrow[dr, "\subseteq"'] & & \mathcal{M}(A) \\
& B \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\subseteq"] \arrow[dr, "\subseteq"'] & & \tilde{A} \arrow[dl, dotted, "\exists"] \\
& B &
\end{tikzcd}
$$

commute. In this sense, $\tilde{A}$ is the minimal and $\mathcal{M}(A)$ is the maximal unitisation.

(iv) Let X be locally compact. Then

$$\mathcal{M}(C_0(X)) = C_b(X) := \{f \colon X \to \mathbb{C} \text{ continuous, bounded}\}$$

and

$$\widetilde{C_0(X)} = C_0(X) \oplus \mathbb{C}1 = C(\hat{X})$$

where $\hat{X}$ is the one point compactification.

(v) Let H be a Hilbert space with $\dim H = \infty$. Then $\mathcal{M}(K(H)) = B(H)$ and

$$\widetilde{K(H)} = K(H) \oplus \mathbb{C}1 \subsetneq B(H)$$

where $1 \in B(H)$.

Remark 2.9: Let A, B be algebras, $\varphi \colon A \to B$ be an algebra homomorphism. Then

$$
\begin{aligned}
\tilde{\varphi} \colon \tilde{A} &\longrightarrow \tilde{B} \\
\lambda 1 + a &\longmapsto \lambda 1 + \varphi(a)
\end{aligned}
$$

is a unital algebra homomorphism with $\tilde{\varphi}|_A = \varphi$. Hence Proposition 2.3 is also true for non-unital C^*-algebras: Let $\varphi \colon A \to B$ as in Proposition 2.3 (iii), extend it to $\tilde{\varphi} \colon \tilde{A} \to \tilde{B}$. Hence $\|\tilde{\varphi}(x)\| \leq \|x\|$ for all $x \in \tilde{A}$, thus $\|\varphi(x)\| \leq x\|$ for all $x \in A$.

Definition 2.10: Let A be a (not necessarily unital) C^*-algebra, $x \in A$. Then we define the *spectrum of x* as

$$\operatorname{Sp}(x) := \begin{cases} \operatorname{Sp}_A x & \text{if } A \text{ is unital,} \\ \operatorname{Sp}_{\tilde{A}}(x) & \text{if } A \text{ is non-unital.} \end{cases}$$

Remark 2.11: (i) If A is non-unital, then $0 \in \operatorname{Sp}(x)$ for all $x \in A$.

(ii) If A is unital and $u \in A$ is unitary (i. e. $u^*u = uu^* = 1$), then we have $\operatorname{Sp}(u) \subseteq \mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \subseteq \mathbb{C}$.

(iii) If $x \in A$ is selfadjoint (i. e. $x = x^*$), then $\operatorname{Sp}(x) \subseteq \mathbb{R}$ (even for $1 \notin A$).

(iv) If $B \subseteq A$ is a C^*-subalgebra and $x \in B$, then $\operatorname{Sp}_B(x) = \operatorname{Sp}_A(x)$.

Definition 2.12: Let A be a commutative C^*-algebra (possibly $1 \notin A$). Then we define the *spectrum of A* as $\operatorname{Spec}(A) := \operatorname{Spec}(\tilde{A}) \setminus \{\tilde{0}\}$ where

$$\begin{aligned} \tilde{0} \colon \tilde{A} &\longrightarrow \mathbb{C} \\ \lambda + x &\longmapsto \lambda \end{aligned}$$

(notice that $\tilde{0}|_A = 0$).

Theorem 2.13: *Let A be a commutative C^*-algebra. Then* $\operatorname{Spec}(A)$ *is locally compact and we have*

$$\begin{tikzcd} \tilde{A} \arrow[r, "\cong"] & C(\operatorname{Spec}(\tilde{A})) \\ A \arrow[u, "\subseteq"'] \arrow[r, "\cong"'] & C_0(\operatorname{Spec}(A)) \arrow[u, "\subseteq"'] \end{tikzcd}$$

Definition 2.14: Let A be a C^*-algebra, $M \subseteq A$ a subset. Then

$$C^*(M) := \text{smallest } C^*\text{-algebra of } A \text{ containing } M$$

is called the *enveloping C^*-algebra of M*.

Remark 2.15: (i) Let A, B be C^*-algebras and $M \subseteq A$. Furthermore let $\varphi, \psi \colon C^*(M) \to B$ be *-homomorphisms. Then it holds: If $\varphi|_M = \psi|_M$, then $\varphi = \psi$ (because $\{x \in C^*(M) \mid \varphi(x) = \psi(x)\} \subseteq A$ is a C^*-algebra containing M).

(ii) $x \in A$ is normal if and only if $x^*x = xx^*$: We know that $C^*(x)$ is commutative if and only if x is normal, as

$$C^*(x) = \operatorname{cl}(\{\text{non-commutative polynomials in } x, x^*\}).$$

Proposition 2.16: *Let A be a C^*-algebra and $x \in A$ be normal.*

(i) *If A is unital, then*

$$\begin{aligned} \operatorname{Spec}(C^*(x,1)) &\xrightarrow{\cong} \operatorname{Sp}(x) \\ \varphi &\longmapsto \varphi(x) \end{aligned}$$

is a homeomorphism.

(ii) *If A is possibly non-unital, $C^*(x)$ non-unital, then*

$$\begin{aligned} \operatorname{Spec}(C^*(x)) &\xrightarrow{\cong} \operatorname{Sp}(x) \setminus \{0\} \\ \varphi &\longmapsto \varphi(x) \end{aligned}$$

is a homeomorphism.

Proof: (i) As in Functional Analysis I: $\varphi \mapsto \varphi(x)$ is injective by Remark 2.15 and surjective, since $\hat{x}(\operatorname{Spec}(C^*(x,1))) = \operatorname{Sp}(x)$; it is continuous with respect to the pointwise topolgy.

(ii) We have

$$\begin{tikzcd} \operatorname{Spec}(C^*(x,1)) \arrow[r, "\cong"] & \operatorname{Sp}(x) \, \cdot \\ \operatorname{Spec}(\widetilde{C^*(x)}) \setminus \{0\} \arrow[u, "\subseteq"'] & \end{tikzcd}$$

∎

Corollary 2.17: *We have the following* functional calculus

(i) *Let A be a unital C^*-algebra and let $x \in A$ be normal. Then*

$$\begin{aligned} \Psi \colon C(\operatorname{Sp}(x)) &\xrightarrow{\cong} C^*(x,1) \\ f &\longmapsto f(x) \end{aligned}$$

is an isometric *-isomorphism mapping $\mathrm{id}_{\operatorname{Sp}(x)} \mapsto x$. Note that $C^*(x,1) \cong C(\operatorname{Spec}(C^*(x,1))) \cong C(\operatorname{Sp}(x))$ by The theorem of Gelfand-Naimark (Reminder 2.4) and Remark 2.15. We have the diagramm

$$\begin{array}{rcccc} \Phi^{-1} \colon C^* & \longrightarrow & C(\operatorname{Spec}(C^*(x,1))) & \longrightarrow & C(\operatorname{Sp}(x)) \\ x & \longmapsto & \hat{x}, \hat{x}(\varphi) = \varphi(x) f & \longmapsto & f \circ \alpha \end{array}$$

where

$$\begin{aligned} \alpha \colon \operatorname{Sp}(x) &\longrightarrow \operatorname{Spec}(C^*(x,1)) \\ \lambda &\longmapsto \varphi, \quad \varphi(x) = \lambda. \end{aligned}$$

(ii) *Let A be non-unital and $x \in A$ be normal. Then we can extend the functional calculus*

$$\Phi : C_0(\mathrm{Sp}(x)) = \{f \in C(\mathrm{Sp}(x)) \mid f(0) = 0\} \longrightarrow C^*(x).$$

Some properties:

- *-homomorphism, so $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $\overline{f}(x) = f(x)^*$,
- $\mathrm{Sp}(f(x)) = f(\mathrm{Sp}(x))$, g continuous on $f(\mathrm{Sp}(x))$ thus $(g \circ f)(x) = g(f(x))$. (see Sheet 3, Exercise 1)

If f is a polynomial, it is clear what $f(x)$ is. If f is a continuous function, that is not a polynomial, think of the Stone-Weierstraß theorem.

Example: (i) Let $x \in A$ be selfadjoint (i. e. $x = x^*$). Then $\mathrm{Sp}(x) \subseteq \mathbb{R}$. Consider $f_+, f_- \colon \mathbb{R} \to [0, \infty)$. Then for $x_+ := f_+(x)$ and $x_- := f_-(x)$ we have that $x = x_+ - x_-$ and x_+, x_- are selfadjoint, too. Furthermore, we know that $f_+(\mathrm{Sp}(x)) = \mathrm{Sp}(x_+)$, $f_-(\mathrm{Sp}(x)) = \mathrm{Sp}(x_-) \subseteq [0, \infty)$ and $x_+x_- = x_-x_+ = 0$.

(ii) Let x be again selfadjoint with $\mathrm{Sp}(x) \subseteq [0, \infty)$. Then there is $\sqrt{x} \in A$ that is selfadjoint with $\mathrm{Sp}(\sqrt{x}) \subseteq [0, \infty)$ and $\sqrt{x}^2 = x$.

Proposition 2.18: *Let A, B be C^*-algebras and let $\varphi \colon A \to B$ be a *-homomorphism, $x \in A$ be normal and $f \in C(\mathrm{Sp}(x))$ (or $f \in C_0(\mathrm{Sp}(x))$). Then*

$$\varphi(f(x)) = f(\varphi(x)).$$

Proof: Since φ is an algebra homomorphism, the statement is clear for polynomials. Then use Stone-Weierstraß approximation. ■

Remark 2.19: This is the *continous* functional calculus (f continuous). There is also a measurable functional calculus (f measurable, von Neumann algebras). Furthermore there is an *analytic*— or *holomorphic* functional calculus, which doesn't require x to be normal.

3 Positive elements in C^*-algebras

Definition 3.1: Let A be a C^*-algebra and $a \in A$. We say that a is *positive* (we write $a \geq 0$), if $a = a^*$ and $\mathrm{Sp}(a) \subseteq [0, \infty)$. We write $a \leq b$, if $b - a \geq 0$.

Remark 3.2: Every positive element in a C^*-algebra possesses a unique positive square root $\sqrt{a} \in A$.

Lemma 3.3: *Let A be a unital C^*-algebra, $a \in A$ be selfadjoint and let $\lambda \geq \|a\|$. Then $a \geq 0$ if and only if $\|\lambda 1 - a\| \leq \lambda$.*

Proof: This statement is clear for functions (sketch missing).

For the fomal proof: $a \geq 0$ holds if and only if $\mathrm{Sp}(a) \subseteq [0, \infty)$ which holds if and only if $\mathrm{id}_{\mathrm{Sp}(a)} \geq 0$. $\mathrm{id}_{\mathrm{Sp}(a)} \geq 0$ holds if and only if

$$\|\lambda 1 - a\| = \|\Phi^{-1}(\lambda 1 - a)\|_\infty = \|\lambda 1 - \mathrm{id}_{\mathrm{Sp}(a)}\|_\infty \leq \lambda. \qquad \blacksquare$$

Proposition 3.4: *If $a, b \geq 0$, then also $a + b \geq 0$.*

Proof: Put $\lambda := \|a\| + \|b\| \geq \|a + b\|$. Then

$$\|\lambda - (a + b)\| \leq \|\|a\|1 + a\| + \|\|b\|1 - b\| \leq \|a\| + \|b\| \leq \lambda,$$

due to $a \geq 0$ and $b \geq 0$ and Lemma 3.3. $\blacksquare$

Lemma 3.5: *Let A be a C^*-algebra, $x \in A$ and $-x^*x \geq 0$. Then $x = 0$.*

Proof: Write $x = x_1 + \mathrm{i}x_2$ where $x_1 = \frac{1}{2}(x + x^*)$ and $x_2 = \frac{1}{2\mathrm{i}}(x - x^*)$, hence x_1 and x_2 are selfadjoint ("decomposition in real— and imaginary part"). We then have

$$x^*x + xx^* = (x_1^2 + \mathrm{i}x_1x_2 - \mathrm{i}x_2x_1 + x_2^2) + (x_1^2 + \mathrm{i}x_2x_1 - \mathrm{i}x_1x_2 + x_2^2) = 2x_1^2 + 2x_2^2,$$

thus $xx^* = 2x_1^2 + 2x_2^2 - x^*x$. Now $2x_1^2$ is positive, because $x_1 = x_1^*$ and for $f(z) := z^2$ we have $\mathrm{Sp}(f(x_1)) = f(\mathrm{Sp}(x_1)) \subseteq [0, \infty)$; the same holds for $2x_2^2$ and $-xx^*$ is positive by assumption. Via Proposition 3.4, xx^* is positive. Now we have

$$\mathrm{Sp}(x^*x) \cup \{0\} = \mathrm{Sp}(xx^*) \cup \{0\} \subseteq [0, \infty), \qquad \text{(refer to Sheet 3, Exercise 2)},$$

on the other hand $\mathrm{Sp}(x^*x) \subseteq (-\infty, 0]$, because $-xx^*$ is positive, thus $\mathrm{Sp}(x^*x) = \{0\}$ and via $\|x^*x\| = r(x^*x) = 0$ we infer that $x = 0$. $\blacksquare$

Proposition 3.6: *Let A be a C^*-algebra and $a \in A$. Then the following are equivalent:*

(i) *a is positive,*

(ii) *There is $h \in A$ with $h = h^*$ such that $a = h^2$,*

(iii) *There is $x \in A$ such that $a = x^*x$.* (algebraic way to express positivity)

Proof: "(i) $\Rightarrow$ (ii)": We define $h := \sqrt{a}$ by Remark 3.2.

"(ii) $\Rightarrow$ (iii)": Put $x := h$.

"(iii) $\Rightarrow$ (i)": Write $x^*x = u - v$, where $u := (x^*x)_+ \geq 0$, $v := (x^*x)_- \geq 0$. For u, v it holds $uv = vu = 0$. Put $y := xv$. Then

$$-y^*y = -vx^*xv = -v(u - v)v = v^3 \geq 0$$

by $f(\mathrm{Sp}(v)) = \mathrm{Sp}(f(v))$. Now from Lemma 3.5 we deduce that $y = 0$, thus $v^3 = 0$ and consequently $v = 0$ (If $\|v^2\|^2 = \|(v^2)^*(v^2)\| = \|v^4\| = 0$, then $v^2 = 0$ and thus $\|v\|^2 = \|v^*v\| = \|v^2\| = 0$). Now it holds $x^*x = u - v = u \geq 0$. ■

Corollary 3.7: *In a C^*-algebra A, we put*

$$A_+ := \{h \in A \mid h = h^*, h \geq 0\} = \{x^*x \mid x \in A\} \subseteq A \qquad \text{(Proposition 3.6)}$$
$$A_{\mathrm{sa}} := \{h \in A \mid h = h^*\} \subsetneq A_+.$$

Then A_+ is a cone, *i. e.*

(i) *If $h + A_+$ and $\lambda \geq 0$, then $\lambda h \in A_+$,*

(ii) *If $h_1, h_2 \in A_+$, then $h_1 + h_2 \in A_+$.*

Moreover $A_+ \cap (-A_+) = \{0\}$ by Lemma 3.5 and Proposition 3.6, $A_{\mathrm{sa}} = A_+ - A_-$ by (Chapter 2) and A_+ is closed by Lemma 3.3.

We observe that due to $\|x^*x\| = \|x\|^2$, the positive elements play a special role in the theory of C^*-algebras.

Proposition 3.8: *In a C^*-algebra, the following holds:*

(i) *If $a, b \in A_{\mathrm{sa}}$, $c \in A$ and $a \leq b$, then $c^*ac \leq c^*bc$.*

(ii) *If $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

(iii) *If A is unital and $0 \leq a \leq b$ are invertible elements of A, then $0 \leq b^{-1} \leq a^{-1}$,*

(iv) *For $\beta \in \mathbb{R}$ with $0 \leq b \leq 1$ and $0 \leq a \leq b$, it holds that $0 \leq a^\beta \leq b^\beta$. In particular we have $0 \leq \sqrt{a} \leq \sqrt{b}$.*

Proof: (i) It holds $c^*bc - c^*ac = c^*(b - a)c = c^*x^*xc \geq 0$ for some $x \in A$.

(ii) Without loss of generality let $1 \in A$. We view a and b as functions $\mathrm{id}_{\mathrm{Sp}(a)}$ and $\mathrm{id}_{\mathrm{Sp}(b)}$, thus

$$\|a\| = \inf\{\lambda \geq 0 \mid \lambda 1 \geq a\}, \qquad \|b\| = \inf\{\lambda \geq 0 \mid \lambda 1 \geq b\}.$$

Now we infer $\|b\|1 \geq b \geq a$, hence $\|b\| \geq \|a\|$.

(iii) Because $a \geq 0$, it also holds $a^{-1} \geq 0$ by the functional calculus. Futhermore the functional calculus gives that if $a \geq 1$, then $a^{-1} \leq 1$. Hence

$$1 = \sqrt{a^{-1}} a \sqrt{a^{-1}} \overset{\text{(i)}}{\leq} \sqrt{a^{-1}} b \sqrt{a^{-1}}$$

and thus $d := \sqrt{a} b^{-1} \sqrt{a} = (\sqrt{a^{-1}} b \sqrt{a^{-1}})^{-1} \leq 1$. Again using (i), we get the estimate $b^{-1} = \sqrt{a^{-1}} d \sqrt{a^{-1}} \leq \sqrt{a^{-1}} 1 \sqrt{a^{-1}} = a^{-1}$.

(iv) This proof is complicated and therefore left out. ■

Remark 3.9: We do not have Proposition 3.8 (iv) for $\beta > 1$, in particular $0 \leq a \leq b$ but $a^2 \not\leq b^2$ may occur. We may even prove

$$\big(\exists \beta > 1 \,\forall x, y \in A : (0 \leq x \leq y \Rightarrow x^\beta \leq y^\beta)\big) \Rightarrow (ab = ba \,\forall a, b \in A).$$

See also Sheet 3, Exercise 4 for other strange things.

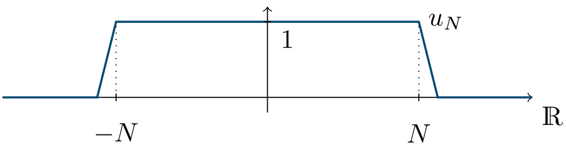
4 Approximate units, ideals, quotients

Definition 4.1: Let X be a topological space. A family $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ is a *net*, if Λ is a filtration (i. e. we have an order "$\leq$" on Λ, such that $\lambda \leq \lambda$ for all $\lambda \in \Lambda$, if $\lambda \leq \mu \leq \lambda$, then $\lambda = \mu$, if $\lambda_1 \leq \lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$ and for all $\lambda, \mu \in \Lambda$, there is $\nu \in \Lambda$ such that $\lambda \leq \nu$ and $\mu \leq \nu$).

$(x_\lambda)_{\lambda\in\Lambda}$ *converges to* $x \in X$, if for any neighbourhood U of x, there is a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Definition 4.2: Let A be a C^*-algebra. An *approximate unit* $(u_\lambda)_{\lambda\in\Lambda}$ in A is a net such that $\lambda \leq \mu \Rightarrow u_\lambda \leq u_\mu$, $0 \leq u_\lambda \leq 1$ (i. e. $\|u_\lambda\| \leq 1$), and $u_\lambda x \to x$ and $x u_\lambda \to x$ for all $x \in A$.

Example 4.3: (i) Let $A = C_0(\mathbb{R})$. Then the net $(u_N)_{N\in\mathbb{N}}$, where u_N is the function



is an approximate unit.

(ii) Let $A = K(H)$ for a separable Hilbert space. Then the net of projections $(u_n)_{n\in\mathbb{N}}$, where $u_n(\sum_{i\in\mathbb{N}} \alpha_i e_i) := \sum_{i=1}^n \alpha_i e_i$ and $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis, is an approximate unit.

Theorem 4.4: *Let A be a C^*-algebra (or an ideal in a C^*-algebra). Then A has an approximate unit.*

Proof: (i) *The set $\Lambda := \{h \in A \mid h \geq 0, \|h\| < 1\}$ is a filtration.*

Proof: The order shall be the order of A_+, $h \leq h$ for all $h \in \Lambda$ is clear. "If $h \leq g \leq h$, then $g = h$" follows from $A_+ \cap -A_+ = \{0\}$. For $f, g, h \in \Lambda$ with $h \leq g \leq f$, $h \leq f$ follows from Proposition 3.4.

Let now $a, b \in \Lambda$. We need to find an element $c \in \Lambda$ such that $a \leq c$ and $b \leq c$. Put $a' := a(1-a)^{-1} \geq 0$ (note that $1 \notin \mathrm{Sp}(a)$, since $\|a\| < 1$). a' is positive because of $t/(1-t) \geq 0$ on $[0,1)$ and the functional calculus.

Using Proposition 3.8 (iii) one can check that if $(1+y)^{-1} \leq (1+x)^{-1}$, then $(x(1+x))^{-1} = 1 - (1+x)^{-1} \leq 1 - (1+y)^{-1} = y(1+y)^{-1}$. Hence

$$a = a'(1+a')^{-1} \leq (a'+b')(1+a'+b')^{-1} =: c.$$

With what we have shown above we know that $a \leq c$ and $b \leq c$. Now $c \geq 0$, since $a' + b' \geq 0$, $1/(1+t) \geq 0$ and $\|c\| < 1$ since $1/(1+t) < 1$, thus $c \in \Lambda$. ■

(ii) *Let $h \geq 0$, $h \in A$ and $n \in \mathbb{N}$. Then $h(\frac{1}{n}+h)^{-1} \in A$, $h(1-h(\frac{1}{n}-h)^{-1}) \leq \frac{1}{n}$.*

Proof: Because of $0 \leq t/(t+\frac{1}{n}) \leq 1$ for all $t \geq 0$ and $t(1-t/(\frac{1}{n}-t)) \leq \frac{1}{n}$ for all $t \geq 0$, the assertion follows from the functional calculus. ■

(iii) *Let $h \geq 0$, $g \in \Lambda$, $h(\frac{1}{n}+h)^{-1} \leq g$. Then $\|h-gh\|^2 \leq \frac{1}{n}\|h\|$ as well as $\|h-hg\|^2 \leq \frac{1}{n}\|h\|$.*

Proof: We have

$$\|h-gh\|^2 = \|h(1-g)^2h\| \leq \|h(1-g)h\| \leq \|h(1-h(\frac{1}{n}-h)^{-1})h\| \leq \frac{1}{n}\|h\|,$$

for the second inequality we use

$$(1-g)-(1-g)^2 = g(1-g) \geq 0 \qquad\qquad (\text{for } 0 \leq g \leq 1)$$

hence $(1-g)^2 \leq (1-g)$, then use Proposition 3.8 (i), (ii). ■

Finally, let $x \in A$ and $\varepsilon > 0$. Put $h := xx^* \geq 0$. By (ii), we know that $\lambda_0 := h(\frac{1}{n}-h)^{-1} \in \Lambda$ for $n \in \mathbb{N}$ with $\frac{1}{n}\|h\| < \varepsilon^2$. Hence for all $g \in \Lambda$ with $\lambda_0 \leq g$:

$$\|x-gx\|^2 = \|(1-g)h(1-g)\| \leq \|h-gh\|(1+\|g\|),$$

if we put $u_g := g$, we conclude $x-gx \to 0$ for $g \to \infty$. Likewise $x-xg \to 0$. ■

Remark 4.5: If A is a separable C^*-algebra (i. e. it has a dense countable subset), then there is an approximate unit $(u_n)_{n\in\mathbb{N}}$, $u_1 \leq u_2 \leq \dots$ (i. e. A is σ-unital, i. e. Λ is countable).

Definition 4.6: Let A be C^*-algebra. I is an *ideal* in A, if $I \subseteq A$ is a closed linear subspace such that AI, $IA \subseteq I$. We write $I \triangleleft A$.

Example 4.7: (i) Let H be a Hilbert space. Then $K(H) \triangleleft B(H)$.

(ii) Let A be a C^*-algebra. Then $A \triangleleft \tilde{A}$ using the notation from Chapter 2.

Lemma 4.8: *Let A be a C^*-algebra.*

(i) *If $I \triangleleft A$, then $I = I^*$ (hence I is a C^*-algebra),*

(ii) *If $I \triangleleft J \triangleleft A$, then $I \triangleleft A$,*

(iii) *If $I \triangleleft A$, $I \subsetneq A$ and $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit of I, then $(u_\lambda)_{\lambda\in\Lambda}$ is not an approximate unit for A*

Proof: (i) Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for I and $x \in I$. Then $u_\lambda x \to x$ and because * is continuous and I is closed, then $I \ni x^*u_\lambda = (u_\lambda x)^* \to x$.

(ii) Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for I, $x \in I$ and $a \in A$. Then $I \ni xu_\lambda a \to xa$, thus $xa \in I$, $ax \in I$.

(iii) Assume $(u_\lambda)_{\lambda\in\Lambda}$ was also an approximate unit for A. Then for all $a \in A$: $I \ni u_\lambda a \to a$, hence $I = A$ which is a contradiction. ■

Theorem 4.9: *Let A be a C^*-algebra and $I \lhd A$. Then A/I is a C^*-algebra.*

Proof: A/I is a Banach algebra with the norm

$$\|\dot{x}\| := \inf\{\|x+z\| \mid z \in I\}$$

for $\dot{x} \in A/I$. It has an involution $(\dot{x})^* := (x^*)^\bullet$ (check that we have "$\dot{x} = \dot{y} \Rightarrow x - y \in I \Rightarrow (x-y)^* \in I \Rightarrow (x^*)^\bullet = (y^*)^\bullet$"). It remains to be shown that $\|\dot{x}^*\dot{x}\| \geq \|\dot{x}\|^2$ (due to Proposition 2.3, "$\leq$" then follows).

Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for I and let $x \in A$. Then $\|\dot{x}\| = \lim\|x - u_\lambda x\|$.

Proof: Let $\varepsilon > 0$. We find $z \in I$ such that $\|x+z\| \leq \|\dot{x}\| + \varepsilon$ and we find $\lambda_0 \in \Lambda$ such that $\|z - u_\lambda z\| < \varepsilon$ for all $\lambda \geq \lambda_0$. Then we have

$$\begin{aligned}\|\dot{x}\| \leq \|x - u_\lambda x\| &\leq \|(1-u_\lambda)(x+z)\| + \|(1-u_\lambda)z\| \\ &\leq \|1-u_\lambda\|\|x+z\| + \|z - u_\lambda z\| \leq \|\dot{x}\| + 2\varepsilon\end{aligned}$$

as $\|x+z\| \leq \|\dot{x}\| + \varepsilon$ and $\|z - u_\lambda z\| < \varepsilon$ by assumption. ■

Now let $x \in A$. Then

$$\begin{aligned}\|\dot{x}\|^2 &= \lim\|x - u_\lambda x\| \\ &= \lim\|(1-u_\lambda)x^*x(1-u_\lambda)\| \\ &= \lim\|(1-u_\lambda)(x^*x+z)(1-u_\lambda)\| \qquad (\text{as } z \in I \text{ and } u_\lambda z - z \to 0) \\ &\leq \|x^*x + z\|.\end{aligned}$$

By taking the Infimum over $z \in I$, we thus get $|\dot{x}\|^2 \leq \|\dot{x}^*\dot{x}\|$. ■

Proposition 4.10: *Let A, B be C^*-algebras and let $\varphi\colon A \to B$ be a *-homomorphism.*

(i) *If φ is injective, then φ is isometric.*

(ii) *$\varphi(A)$ is a C^*-algebra and $A/\ker(\varphi) \cong \varphi(A)$.*

Proof: (i) We need to show that $\|\varphi(x^*x)\| = \|x^*x\|$, then $\|\varphi(x)\|^2 = \|\varphi(x^*x)\| = \|x^*x\| = \|x\|^2$. Let's assume that φ is isometric, i. e. $\|\varphi(x^*x)\| \leq \|x^*x\|$ for some $x \in A$. Consider $f\colon \mathbb{R} \to \mathbb{R}$ with $0 \leq f \leq 1$, $f|_{(\|x^*x\|,\infty)} = 1$ and $f|_{[0,\|\varphi(x^*x)\|]} \equiv 0$ (refer to Figure 4.1). Then

$$\varphi(f(x^*x)) = f(\varphi(x^*x)) = 0$$

where $0 \neq \|f(x^*x)\| = \|f|_{[0,\|x^*x\|]}\|_\infty = 1$, since $r(x^*x) = \|x^*x\|$, hence $\|x^*x\| \in \mathrm{Sp}(x^*x) \subseteq [0,\infty)$ (in principle: $\|x^*x\| \in \mathrm{Sp}(x^*x)$ or $-\|x^*x\| \in \mathrm{Sp}(x^*x)$) and $\|f(\varphi(x^*x))\| = \|f|_{[0,\|\varphi(x^*x)\|]}\|_\infty = 0$, thus φ is not injective.

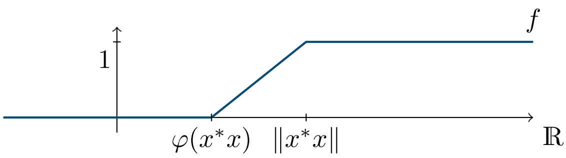


Figure 4.1: Sketch of the function f as mentioned in part (i) of the proof.

(ii) We have the diagramm

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & B \\
& A/\ker(\varphi) \arrow[ur, "\exists\,\dot{\varphi}"'] &
\end{tikzcd}$$

where $\dot{\varphi}(\dot{x}) := \varphi(x)$ is a well-defined *-homomorphism. Note that $\ker(\varphi) \triangleleft A$. By part (i), $\dot{\varphi}$ is isometric, therefore $\varphi(A) = \dot{\varphi}(A/\ker(\varphi))$ is complete and thus a C^*-algebra. ■

Remark 4.11 (Homological properties of C^*-algebras): The sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is exact if and only if $I \triangleleft A$ and $B \cong A/I$.

5 Positive linear functionals and the GNS construction

Definition 5.1: Let A be a C^*-algebra and $\varphi\colon A \to \mathbb{C}$ be a linear functional. φ is called *positive* (in signs $\varphi \geq 0$), if $\varphi(x) \geq 0$ for all $x \geq 0$.

A positive linear functional preserves the order, i. e. "$x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$" holds.

Example 5.2: (i) Let $A = C([0,1])$. Then $\varphi_t(f) := f(t)$ or $\varphi(f) := \int_0^1 f(t)\,dt$ are positive functionals. More generally, we have the correspondence:

$$\begin{aligned}\{\text{positive functionals on } A\} &\xrightarrow{\sim} \{\text{Radon measures on } [0,1]\}\\ \varphi &\longmapsto \mu \text{ with } \varphi(f) = \textstyle\int_0^1 f\,d\mu\,.\end{aligned}$$

(ii) Let $A = M_n(\mathbb{C})$. Then

$$\begin{aligned}\mathrm{tr}\colon A &\longrightarrow \mathbb{C}\\ (a_{i,j})_{1\leq i,j\leq n} &\longmapsto \textstyle\sum_{i=1}^n a_{i,i}\end{aligned}$$

is a positive linear functional.

(iii) Let $A = B(H)$ and $\xi \in H$. Then $\varphi_\xi(x) := \langle x\xi, \xi\rangle$ is a positive functional.

Lemma 5.3: *Let A be a C^*-algebra and $\varphi\colon A \to \mathbb{C}$ be positive. Then $\langle x, y\rangle := \varphi(y^*x)$ is a positive sesquilinear form.*

Proposition 5.4: *Let A be a C^*-algebra, $\varphi\colon A \to \mathbb{C}$ be positive.*

(i) *φ is bounded (hence continuous),*

(ii) *φ is involutive, i. e. $\varphi(x^*) = \overline{\varphi(x)}$ and $|\varphi(x)|^2 \leq \|\varphi\|\varphi(x^*x)$ for all $x \in A$.*

Proof: (i) (1) *φ is bounded on $S := \{x \in A \mid x \geq 0, \|x\| \leq 1\}$.*

Proof: Assume φ was not bounded on S. Then there was a sequence $(a_n)_{n\in\mathbb{N}} \subseteq S$ such that $\varphi(a_n) \geq 2^n$. Then $a := \sum_{n\in\mathbb{N}} \frac{1}{2^n} a_n \geq 0$ by (Corollary 3.7), but

$$\varphi(a) \geq \varphi\Big(\sum_{n=1}^N \frac{1}{2^n} a_n\Big) \geq N$$

which was a contradiction. ■

(2) For an arbitrary $z \in A$ write $z = (\mathrm{Re}(z))_+ - (\mathrm{Re}(z)_- + \mathrm{i}\big((\mathrm{Im}(z))_+ - (\mathrm{Im}(z))_-\big)$ as a linear combination of four positive elements with norm smaller or equal to the norm of z. Thus $\|\varphi(z)\| \leq 4K\|z\|$, where K is the bound from (1).

(ii) By (Lemma 5.3), $\langle\cdot,\cdot\rangle$ is a positive sesquilinear form, hence the polarisation identity is satisfied (Theorem 5.7, Funtional Analaysis I). Hence $\langle x,y\rangle=\overline{\langle y,x\rangle}$. Thus

$$\varphi(x^*)\leftarrow\varphi(x^*u_\lambda)=\langle u_\lambda,x\rangle=\overline{\langle x,u_\lambda\rangle}=\overline{\varphi(u_\lambda^*x)}=\overline{\varphi(u,x)}\to\overline{\varphi(x)},$$

and moreover

$$|\varphi(x)|^2\leftarrow|\varphi(u_\lambda x)|^2=|\langle x,u_\lambda\rangle|^2\leq\langle x,x\rangle\langle u_\lambda,u_\lambda\rangle=\varphi(x^*x)\varphi(u_\lambda^2).$$ ■

Proposition 5.5: *Let A be a C^*-algebra and $\varphi\colon A\to\mathbb{C}$ be continuous and linear. Then the following are equivalent:*

(i) *φ is positive,*

(ii) *For all approximate units $(u_\lambda)_{\lambda\in\Lambda}\subseteq A$ it holds $\|\varphi\|=\lim\varphi(u_\lambda)$,*

(iii) *There is an approximate unit $(u_\lambda)_{\lambda\in\Lambda}\subseteq A$ such that $\|\varphi\|=\lim\varphi(u_\lambda)$.*

Proof: "(i) $\Rightarrow$ (ii)": Without loss of generality, let $\|\varphi\|=1$. Then $(\varphi(u_\lambda))_{\lambda\in\Lambda}\subseteq\mathbb{C}$ is a bounded, monontonically increasing net in $\mathbb{R}_+$, hence $\varphi(u_\lambda)\uparrow\alpha\leq1$ for some $\alpha\in\mathbb{R}$. For $x\in A$, $\|x\|\leq1$ we have

$$|\varphi(x)|^2\leftarrow|\varphi(u_\lambda x)|^2\leq\varphi(u_\lambda^2)\varphi(x^*x)\leq\varphi(u_\lambda)\varphi(x^*x)\leq\alpha$$

because of $0\leq u_\lambda\leq u_\lambda^2$ and the functional calculus. Since $\|\varphi\|=1$ we find $x_n\in A$, $\|x_n\|\leq1$ such that $\alpha\geq|\varphi(x_n)|^2\to1$, thus $\alpha=1=\|\varphi\|$. "(ii) $\Rightarrow$ (iii)" is obviously true.

"(iii) $\Rightarrow$ (i)": Without loss of generality, let $\|\varphi\|=1$, thus $\varphi(u_\lambda)\to1$.

(1) *For $x\in A$ selfadjoint, $\|x\|\leq1$, we have $\varphi(x)\in\mathbb{R}$.*

Proof: Let $\varphi(x)=\alpha+\mathrm{i}\beta$, without loss of generality $\beta\leq0$. Assume $\beta<0$. Then for all $n\in\mathbb{N}$ we have

$$\begin{aligned}\|x-\mathrm{i}nu_\lambda\|^2=\|(x+\mathrm{i}nu_\lambda)(x-\mathrm{i}nu_\lambda)\|&=\|x^2+n^2u_\lambda^2-\mathrm{i}n(xu_\lambda u_\lambda x)\|\\&\leq1+n^2+n\|xu_\lambda-u_\lambda x.\end{aligned}$$

Therefore, $|\beta|^2+2|\beta|n+n^2=|\mathrm{Im}(\varphi(x)-\mathrm{i}n)|^2\leq|\varphi(x)-\mathrm{i}n|^2\leftarrow|\varphi(x-\mathrm{i}nu_\lambda)|$. Using the result of the above calulcation, we obtain

$$|\beta|^2+2|\beta|n+n^2\leq1+n^2+n\|xu_\lambda-u_\lambda x\|,$$

thus $2|\beta|n\leq1-|\beta|^2$, hence $\beta=0$, which is a contradiction. ■

(2) Let now $x\geq0$, $\|x\|\leq1$. Then $-1\leq u_\lambda-x\leq1$ (where the last inequality holds because we have "$u_\lambda\leq1\Rightarrow1-u_\lambda\geq0\Rightarrow1-u_\lambda+x\geq0$"), hence $\|u_\lambda-x\|\leq1$. By (i) we thus have

$$1-\varphi(x)\leq|1-\varphi(x)|\leftarrow|\varphi(u_\lambda-1)|\leq1$$

which implies $\varphi(x)\geq0$. ■

Corollary 5.6: *Let A be a unital C^*-algebra, $\varphi\colon A \to \mathbb{C}$ be continuous and linear. φ is positive if and only if $\varphi(1) = \|\varphi\|$.*

Corollary 5.7: *Let φ, φ' be two positive functional. Then $\|\varphi + \varphi'\| = \|\varphi\| + \|\varphi'\|$.*

Proof: We have

$$\|\varphi\| + \|\varphi'\| \leftarrow \varphi(u_\lambda) + \varphi'(u_\lambda) = (\varphi + \varphi')(u_\lambda) \to \|\varphi + \varphi'\|. \qquad \blacksquare$$

Definition 5.8: A *state* on a C^*-algebra is a positive linear functional φ with $\|\varphi\| = 1$.

Remark 5.9: Let A be a unital C^*-algebra. φ is a state on A if and only if φ is positive and $\varphi(1) = 1$.

Reminder 5.10 (Theorem of Hahn-Banach[1]): Let E be a normed $\mathbb{C}$-vector space, $F \subseteq E$ be a linear subspace and $f\colon F \to \mathbb{C}$ continuous and linear. Then there is continuous and linear $\tilde{f}\colon E \to \mathbb{C}$ such that $\|\tilde{f}\| = \|f\|$ and $\tilde{f}|_F = f$.

Theorem 5.11: *Let A be a C^*-algebra, $x \in A$ be normal. Then we find a state $\varphi\colon A \to \mathbb{C}$ such that $|\varphi(x)| = \|x\|$.*

Proof: By the Gelfand isomorphism we find a character $\varphi_0\colon C^*(x, 1) \to \mathbb{C}$ such that $\varphi_0(1) = 1$ with $|\varphi_0(x)| = |\hat{x}(\varphi_0)| = \|\hat{x}\|_\infty = \|x\|$. In particular, φ_0 is linear and continuous ($\|\varphi_0\| = 1$). By the Hahn-Banach theorem (Reminder 5.10), we find an extension $\tilde{\varphi}_0\colon \tilde{A} \to \mathbb{C}$ that is linear and continuous and fulfills that $\|\varphi_0\| = \|\tilde{\varphi}_0\| = 1 = \varphi_0(1) = \tilde{\varphi}_0(1)$.

By Corollary 5.6 we know that $\tilde{\varphi}_0 \geq 0$ and thus $\varphi := \tilde{\varphi}_0|_A$ is positive and it holds that $|\varphi(x)| = \|x\|$, $\|\varphi\| = 1$. $\blacksquare$

Definition 5.12: Let H, H_1, H_2, $(H_i)_{i\in I}$ be Hilbert spaces and A be a C^*-algebra.

(i) A *representation* of A on H is a *-homomorphism $\pi\colon A \to B(H)$.

(ii) Two representations (π_1, H_1), (π_2, H_2) are equivalent, if there is a unitary map $U\colon H_1 \to H_2$ such that $\pi_2(x) = U_{\pi_1}(x)U^*$ for all $x \in A$.

(iii) Let $(\pi_i, H_i)_{i\in I}$ be representations of A. Then $(\bigoplus_{i\in I} \pi_i, \bigoplus_{i\in I} H_i)$ is a representation on $\bigoplus_{i\in I} H_i$ given by

$$\Big(\bigoplus_{i\in I} \pi_i\Big)(x)\xi_j := \pi_j(x)\xi_j \qquad\qquad j \in J.$$

(iv) A representation π is *non-degenerate* if $\mathrm{cl}(\pi(A)H) = H$.

(v) A representation π is *cyclic* if there is $\xi \in H$ (the so called *cyclic vector*) such that $\mathrm{cl}(\pi(A)\xi) = H$.

[1] Refer to the Functional Analysis I lecture notes, Chapter 2.

Every cyclic representation is in particular non-degenerate.

Remark 5.13: (i) Every representation is a direct sum of a non-degenerate and a zero representation.

(ii) Every non-degenerate representation is a direct sum of cyclic representations.

Lemma 5.14: *Let (π_1, H_1, ξ_1) and (π_2, H_2, ξ_2) be two cyclic representations and $f_1, f_2 \colon A \to \mathbb{C}$ be positive linear functionals with $f_i(x) = \langle \pi_i(x)\xi_i, \xi_i \rangle$ for $i = 1, 2$. If $f_1 = f_2$ holds, then there is a unitary $U \colon H_1 \to H_2$ with $U\xi_1 = \xi_2$ such that $\pi_2(x) = U\pi_1(x)U^*$.*

Proof: See Sheet 4, Exercise 3. ■

Theorem 5.15 (GNS-Construction): *Let f be a state on a C^*-algebra A. Then there is a unique (up to unitary equivalence) cyclic representation (π_f, H_f, ξ_f) such that $f(x) = \langle \pi_f(x)\xi_f, \xi_f \rangle$.*

Proof: (1) Firstly, given the data A and f, we want to construct the Hilbert space for the representation of A:

- $\langle x, y \rangle_f := f(y^*x)$ is a positive sesquilinear form on $A \times A$
- Put $\mathcal{N}_f := \{x \in A \mid \langle x, x \rangle_f = 0\}$ and define $K_f := A/\mathcal{N}_f$.

Then K_f is a pre-Hilbert space: For the quotient map $\gamma \colon A \to A/\mathcal{N}_f$, the expression $\langle \gamma(x), \gamma(y) \rangle := \langle x, y \rangle_f$ is well-defined and γ continuous because

$$\|\gamma(x)\|^2 = \langle \gamma(x), \gamma(x) \rangle = f(x^*x) \leq \|x\|^2$$

- Put $H_f := \mathrm{cl}_{\|\cdot\|_f}(K_f)$. H_f is then a Hilbert space with the inner product $\langle \gamma(x), \gamma(y) \rangle = f(y^*x)$.

(2) Secondly, we want to construct the representation of A on H_f:

- Define

$$\begin{aligned} \pi_f^0(x) \colon K_f &\longrightarrow H_f \\ \gamma(y) &\longmapsto \gamma(x, y). \end{aligned}$$

  Then π_f^0 is continuous:

$$\|\pi_f^0(x)\gamma(y)\|^2 = \|\gamma(xy)\|^2 = f(y^*x^*xy) \leq \|x^*x\| f(y^*y) = \|x\|^2 \|\gamma(y)|^2,$$

  and thus $\|\pi_f^0(x)\| \leq \|x\|$. Also, this proves thast $\pi_f^0(x)$ is well-defined (If $\gamma(y) = \gamma(z)$, then $\|\gamma(xy) - \gamma(xz)\|^2 = \|\gamma(x(y-z))\|^2 \leq \|x\|^2 \|\gamma(y-z)\|^2 = 0$).
- Extend π_f^0 to $\pi_f(x) \colon H_f \to H_f$, then it holds $\|\pi_f(x)\| \leq \|x\|$ and due to $\pi_f^0(x)\pi_f^0(y) = \pi_f^0(xy)$, we also have $\pi_f(x)\pi_f(y) = \pi_f(xy)$.

- π_f^0 is a *-homomorphism, because

$$\begin{aligned}\pi_f^0(x^*) = \langle \pi_f(x)\gamma(y), \gamma(z)\rangle = f(z^*xy) = f((x^*z)^*y)\\ = \langle \gamma(y), \gamma(x^*z)\rangle = \pi_f^0(x^*)\gamma(z) = \pi_f^0(x)^*.\end{aligned}$$

  This property transfers to the limit, thus π_f is a *-homomorphism.

(3) Thirdly, we want to construct the cyclic vector ξ_f:

- Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Put $\xi_f := \lim \gamma(u_\lambda)$. The limit $\lim \gamma(u_\lambda)$ exists: Let $\lambda \geq \mu$. Then

$$\|\gamma(u_\lambda) - \gamma(u_\lambda)\|^2 = f((u_\lambda - u_\mu))^2) \leq f(u_\lambda - u_\mu) < \varepsilon$$

  for λ, μ large, as we have $u_\lambda \geq u_\mu$, $1 \leq u_\lambda \geq u_\lambda - u_\mu \geq 0$ and via the functional calculus, we get $(u_\lambda - u_\mu)^2 \leq (u_\lambda - u_\mu)^2$ and thus $(\gamma(u_\lambda))_{\lambda\in\Lambda}$ is a Cauchy-net.
- ξ_f is cyclic, as for $x \in A$ we have $\pi_f(x)\gamma(u_\lambda) = \gamma(xu_\lambda) \to \gamma(x)$, hence $\pi_f(x)\xi_f = \gamma(x)$ for all $\gamma(x) \in K_f$.

(4) Fourthly, we have

$$\langle \pi_f(x)\xi_f, \xi_f\rangle = \lim\langle \pi_f(x)\gamma(u_\lambda), \gamma(u_\lambda)\rangle = \lim f(u_\lambda x u_\lambda) = f(x).$$

(5) Finally, the uniqueness follows from Lemma 5.14. ■

Corollary 5.16 (Second Fundamental Theorem of C^*-algebras): *Every C^*-algebra A admits a faithful (i. e. injective) representation $\pi\colon A \hookrightarrow B(H)$. Hence, A is isomorphic to a C^*-subalgebra of $B(H)$.*

Of the above Corollary we can make "Every (abstractly defined) C^*-algebra has a concrete representation." or "The abstract (Chapter 2) and the concrete definition (as $\|\cdot\|$-closed *-subalgebra of $B(H)$ for a Hilbert space H) coincide."

Proof: Put $\pi := \bigoplus_{f \text{ state on } A} \pi_f$. We now need to show, that π is faithful; let therefore $x \in A\setminus\{0\}$. By (Theorem 5.11) we find a state f such that $f(x^*x) = \|x\|^2$. By Theorem 5.15, we get that $\|\pi_f(x)\xi_f\|^2 = f(x^*x) = \|x\|^2 \neq 0$, so $\pi_f(x) \neq 0$ and thus $\pi(x) \neq 0$ and π is faithful. ■

Remark 5.17: If A is separable, then H in Corollary 5.16 may be chosen to be separable.

Proof: Let $\{x_1, x_2, \dots, \} \subseteq A$ be dense and countable. We then only need to check that $f_n(x_n^*x_n) = \|x_n\|^2$. Then put $\pi := \bigoplus_{n\in\mathbb{N}} \pi f_n$. ■

Remark 5.18: The GNS-Construction (Theorem 5.15) using pure states to irreducible representations.

(i) A representation $\pi\colon A \to H$ is called *irreducible*, if one of the following conditions is satisfied:

(1) $\pi = 0$,

(2) $\dim H = 1$,

(3) The only closed subspaces K of H such that $\pi(A)K \subseteq K$ (in this case it also holds $\pi(A)K^\perp \subseteq K^\perp$) are 0 and H.

(4) $\pi(A)' = \mathbb{C}1$, where $\pi(A)' := \{x \in B(H) \mid x\pi(y) = \pi(y)x \,\forall\, y \in A\}$ is the so called *commutant*.

(5) Every vector $0 \neq \xi \in H$ is cyclic.

(3) - (5) are equivalent conditions.

(ii) If $\dim H < \infty$, then $\pi = \pi_1 \oplus \cdots \oplus \pi_n$, where the π_i $(1 \leq i \leq n)$ are irriducible.

(iii) $f\colon A \to \mathbb{C}$ is called a *pure state*, if it holds: "$0 \leq g \leq f \Rightarrow \exists\, \lambda \in [0,1] : g = \lambda f$". If f is a pure state, then (π_f, H_f, ξ_f) is irreducible.

(iv) For all $0 \neq x \in A$ there is an irriducible representation $\pi\colon A \to H$ such that $\|\pi(x)\| = \|x\|$.

6 von Neumann algebras

Motivation 6.1: In Definition 1.1 we have introduced several locally convex topologies on $B(H)$. A *-subalgebra $A \subseteq B(H)$ that is closed with respect to the operator norm topology is a C^*-algebra; in fact, each C^*-algebra arises in this way for a suitable Hilbert space H (see Corollary 5.16). What happens, if A is instead required to be closed with respect to the weak— or the strong operator topology?

Let in the following $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space.

Lemma 6.2: *Let $\varphi\colon B(H) \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

(i) *φ is continuous with respect to the weak operator topology.*

(ii) *φ is continuous with respect to the strong operator topology.*

(iii) *There is $n \in \mathbb{N}$ and there are $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$ such that*

$$\varphi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle \qquad \text{for all } x \in H.$$

Proof: Exercise! ■

Theorem 6.3: *Let $C \subseteq B(H)$ be a convex set. Then C is weakly closed if and only if C is strongly closed.*

The proof of this theorem relies on Lemma 6.2 and the following variant of the Hahn-Banach separation theorem, which can be derived from Theorem 2.7 from the Functional Analysis I lecture notes.

Theorem 6.4 (Hahn-Banach): *Let $(X, \mathfrak{T})$ be a locally convex topological $\mathbb{C}$-vector space. Suppose that $C \subseteq X$ is a closed convex subset and $x_0 \in X \setminus C$. Then there exists $\varphi \in (X, \mathfrak{T})^*$*[1] *and $\gamma \in \mathbb{R}$ such that*

$$\operatorname{Re}(\varphi(x)) < \gamma < \operatorname{Re}(\varphi(x_0)) \qquad \text{for all } x \in C.$$

Proof (of Theorem 6.3): "$\Rightarrow$" is clear, since $\mathfrak{T}_{\mathrm{WOT}} \subseteq \mathfrak{T}_{\mathrm{SOT}}$ (see (Remark 1.2)).
"$\Leftarrow$": Suppose that C is strongly closed. We assume $C \subsetneq \mathrm{cl}_{\mathrm{WOT}}(C)$, i. e. there is $x_0 \in \mathrm{cl}_{\mathrm{WOT}}(C) \setminus C$. Thus by Theorem 6.4, we find $\varphi \in (B(H), \mathfrak{T}_{\mathrm{SOT}})^*$ and $\gamma \in \mathbb{R}$, such that

$$\operatorname{Re}(\varphi(x)) < \gamma < \operatorname{Re}(\varphi(x_0)) \qquad \text{for all } x \in C.$$

[1] In the following we will denote with $(X, \mathfrak{T})^*$ the set

$$\{f : X \to \mathbb{C} \mid f \text{ continuous with respect to } \mathfrak{T}\}.$$

Due to Lemma 6.2, φ is also weakly continuous and thus we infer that

$$\mathrm{Re}(\varphi(x)) < \gamma < \mathrm{Re}(\varphi(x_0)) \qquad \text{for all } x \in \mathrm{cl}_{\mathrm{WOT}}(C),$$

which contradicts $x_0 \in \mathrm{cl}_{\mathrm{WOT}}(C)$. Hence $C = \mathrm{cl}_{\mathrm{WOT}}(C)$, i. e. C is weakly closed. ■

Recall that in any topological vector space, the closure of a convex set is again convex. Thus, we deduce from Theorem 6.3:

Corollary 6.5: *Let $C \subseteq B(H)$ be a convex set. Then $\mathrm{cl}_{\mathrm{WOT}}(C) = \mathrm{cl}_{\mathrm{SOT}}(C)$.*

Proof: • $\mathrm{cl}_{\mathrm{WOT}}(C)$ is convex and weakly closed, thus by Theorem 6.3 it holds that $\mathrm{cl}_{\mathrm{WOT}}(C)$ is strongly closed and thus $\mathrm{cl}_{\mathrm{WOT}}(C) \supseteq \mathrm{cl}_{\mathrm{SOT}}(C)$.

• $\mathrm{cl}_{\mathrm{SOT}}(C)$ is convex and strongly closed, thus by Theorem 6.3 it holds that $\mathrm{cl}_{\mathrm{SOT}}(C)$ is weakly closed and thus $\mathrm{cl}_{\mathrm{SOT}}(C) \supseteq \mathrm{cl}_{\mathrm{WOT}}(C)$. ■

Thus, since each *-subalgebra A of $B(H)$ is in particular convex, we see that $\mathrm{cl}_{\mathrm{WOT}}(A) = \mathrm{cl}_{\mathrm{SOT}}(A)$. For unital A, i. e. $1 = \mathrm{id}_H \in A$, we can handle this huge analytic object by purely algebraic means.

Definition 6.6: Let $S \subseteq B(H)$ be any subset. We call

(i) $S' := \{y \in B(H) \mid \forall x \in S : xy = yx\}$ the *commutant of S*,

(ii) $S'' := (S')'$ the *bicommutant*— or *double commutant of S*.

Lemma 6.7: *Let $S \subseteq B(H)$ be any subset.*

(i) *S' is a weakly (and strongly) closed unital subalgebra of $B(H)$.*

(ii) *If $S = S^*$ (i. e. if $x \in S$, then $x^* \in S$), then S' is a weakly closed (and strongly) closed unital *-subalgebra of $B(H)$.*

(iii) *We have $S \subseteq S''$ and $S''' := (S'')' = S'$. If $T \subseteq B(H)$ is another subset, then*

$$S \subseteq T \quad \Longrightarrow \quad T' \subseteq S'.$$

Proof: Exercise! ■

Lemma 6.7 tells us that each *-algebra $A \subseteq B(H)$ sits inside the weakly (and strongly) closed unital *-subalgebra $A'' \subseteq B(H)$; therefore

$$A \subseteq \mathrm{cl}_{\mathrm{SOT}}(A) \subseteq \mathrm{cl}_{\mathrm{WOT}}(A) \subseteq A''.$$

We can say more, if A is unital.

Theorem 6.8 (von Neumann's bicommutant theorem): *Let $M \subseteq B(H)$ be a unital *-subalgebra. Then the following statements are equivalent:*

(i) $M = M''$ (algebraic condition)

(ii) *M is weakly closed.* (analytic condition)

(iii) *M is strongly closed.* (analytic condition)

Definition 6.9: Let $M \subseteq B(H)$ be a unital *-subalgebra. If M satisfies the equivalent conditions in Theorem 6.8, then M is called a *von Neumann algebra* (acting on H).

The theory of von Neumann algebras began with a series of groundbreaking papers "On rings of operators" by Francis J. Murray and John von Neumann that appeared 1936 to 1943.

Theorem 6.10: *Let $A \subseteq B(H)$ be a unital *-subalgebra. Then $A'' \subseteq \mathrm{cl}_{\mathrm{SOT}}(A)$ and thus (Lemma 6.7) we have the equalities*

$$\mathrm{cl}_{\mathrm{SOT}}(A) = \mathrm{cl}_{\mathrm{WOT}}(A) = A''.$$

Proof: Take $y \in A''$. We have to show that $y \in \mathrm{cl}_{\mathrm{SOT}}(A)$, which means that

$$\forall\, \varepsilon > 0,\ \xi_1, \ldots, \xi_n \in H\, \exists\, x \in A : \|y\xi_i - x\xi_i\| < \varepsilon \qquad \forall\, i = 1, \ldots, n.$$

① Consider the case $n = 1$. Let $\varepsilon > 0$ and $\xi \in H$ be be given. Put

$$H_0 := \mathrm{cl}(A\xi) := \mathrm{cl}(\{x\xi \mid x \in A\}) \subseteq H$$

and let p be the orthogonal projection onto H_0. Then $p \in A'$: We clearly have $aH_0 \subseteq H_0$ for all $a \in A$ and also $aH_0^\perp \subseteq H_0^\perp$ for all $a \in A$, since for $\zeta \in H_0^\perp$, $\eta \in H$ and $a \in A$ we have

$$\langle a\zeta, \eta\rangle = \langle \zeta, a^*\eta\rangle = 0.$$

Thus for all $\zeta \in H_0^\perp$, $\eta \in H_0$ and $a \in A$, we get that

$$ap(\eta + \zeta) = a\eta = p(a\eta) = p(a\eta + a\zeta) = pa(\eta + \zeta),$$

thus $ap = pa$ for all $a \in A$.

Now, since $y \in A''$, we see that $py = yp$, which implies

$$yH_0 = ypH_0 = pyH_0 \subseteq H_0.$$

Because $1 \in A$ implies $\xi = 1\xi \in H_0$, we get $y\xi_0 \in H_0 = \mathrm{cl}(A\xi)$, i.e. there is $x \in A$ such that $\|y\xi - x\xi\| < \varepsilon$.

② The case of n vectors $\xi_1, \ldots, \xi_n \in H$ will be reduced to ① by a *matrix trick*. Consider $H^n = \bigoplus_{i=1}^n H \ni \xi := (\xi_1, \ldots, \xi_n)^\top$ and a mapping

$$\begin{aligned} \pi \colon A &\longrightarrow B(H)^n \\ a &\longmapsto \pi(a), \end{aligned}$$

where $\pi(a) = \operatorname{diag}(a, \dots, a)$, i. e. $\pi(a)(\eta_1, \dots, \eta_n)^\top := (a\eta_1, \dots, a\eta_1)^\top$. Then $\pi(A)$ is a unital *-subalgebra of $B(H^n)$ and we have $\pi(y) \in \pi(A)''$; note that $\pi(A)' = \{(a_{i,j})_{1\le i,j\le n} \mid \forall\, i,j = 1, \dots, n : a_{i,j} \in A'\}$.

Thus, applying ① to $\pi(y) \in \pi(A)''$ and $\xi \in H^n$ yields $x \in A$ such that

$$\|\pi(y)\xi - \pi(x)\xi\| < \varepsilon,$$

i. e. $\|y\xi_i - x\xi_i\| < \varepsilon$ for $i = 1, \dots, n$. ■

Proof (of Theorem 6.8): "(i) $\Rightarrow$ (ii)": is the statement that was shown in Lemma 6.7, "(ii) $\Rightarrow$ (iii)" is a consequence of $\mathfrak{T}_{\mathrm{WOT}} \subseteq \mathfrak{T}_{\mathrm{SOT}}$ (as stated in Remark 1.2) and "(iii) $\Rightarrow$ (i)" is the statement Theorem 6.10. ■

Definition 6.11: Let $S \subseteq B(H)$ be any subset. Put $S^* := \{x^* \mid x \in S\}$. Then $\mathrm{vN}(S) := (S \cup S^*)''$ is called the *von Neumman algebra generated by S*; due to Exercise 1 (d), Sheet 5, this is the smallest von Neumann algebra $M \subseteq B(H)$ that contains S.

Remark 6.12: (i) Every von Neumann algebra $M \subseteq B(H)$ is also closed with respect to the operator norm topology (see Remark 1.2) and thus a C^*-algebra. Their general theories, however, are wildly different.

(ii) Obvious examples of von Neumann algebras are $B(H)$ and $\mathbb{C}1 \subseteq B(H)$. It is less obvious that there are other non-trivial examples.

(iii) For any subset $S = S^* \subseteq B(H)$, S' is a von Neumann algebra.

(iv) Von Neumann algebras are closed under the measurable functional calculus (see Theorem 11.5 in the Functional Analysis I lecture notes): If $M \subseteq B(H)$ is a von Neumann algebra and $x \in M$ a normal operator, then the continuous functional calculus

$$\Phi\colon C(\mathrm{Sp}(x)) \longrightarrow C^*(x, 1) \subseteq B(H)$$

admits a unique extension

$$\tilde\Phi\colon B_b(\mathrm{Sp}(x)) \longrightarrow W^*(x, 1) = \mathrm{vN}(x) \subseteq M \subseteq B(H)$$

such that $\tilde\Phi$ is a *-homomorphism with

- $\|\tilde\Phi(f)\| \le \|f\|_\infty$ for all $f \in B_b(\mathrm{Sp}(x))$,
- ($f_n \to f$ pointwise, f bounded) $\Rightarrow$ ($\tilde\Phi(f_n) \to \tilde\Phi(f)$ in WOT).

In particular, we know that all spectral projections $E_x(B) = \tilde\Phi(\chi_B)$ for Borel subsets $B \subseteq \mathrm{Sp}(x)$ belong to $\mathrm{vN}(x)$ and hence to M. Therefore, von Neumann algebras contain — in contrast to C^*-algebras — many projections; note that there are C^*-algebras that contain no projections except 0 and 1.

It follows that $\mathrm{cl}_{\mathrm{ONT}}(\langle\{p \in M \mid p \text{ projection}\}\rangle) = M$: Indeed, if $x \in M$ is selfadjoint, we may approximate $\mathrm{id}_{\mathrm{Sp}(x)} \in C(\mathrm{Sp}(x))$ uniformly by step functions.

Lemma 6.13: *Let $M \subseteq B(H)$ be a von Neumann algebra and $x \in B(H)$. Then $x \in M$ holds if and only if $ux = xu$ for all unitary $u \in M'$.*

Proof: "$\Rightarrow$": This is trivial, since $M = M''$.
"$\Leftarrow$": Every element in a C^*-algebra C is a linear combination of (at most four) unitaries.

Proof: Let $a = a^* \in C$ with $\|a\| \leq 1$. Then we have the decomposition $a = \frac{1}{2}(u+u^*)$ where $u := a + \mathrm{i}\sqrt{1-a^2}$. Each a can be written as $a = \mathrm{Re}(a) + \mathrm{i}\mathrm{Im}(a)$ where

$$\mathrm{Re}(a) = \frac{1}{2}(a + a^*) \quad \text{and} \quad \mathrm{Im}(a) = \frac{1}{2\mathrm{i}}(a - a^*). \qquad \blacksquare$$

Applying this to $M = M'$, yields that $yx = xy$ holds for all $y \in M'$, thus $x \in (M')' = M'' = M$. ■

Corollary 6.14 (Polar decomposition in von Neumann algebras): *Let $M \subseteq B(H)$ be a von Neumann algebra and $x \in M$. Consider the polar decomposition $x = v|x|$, $|x| = (x^*x)^{1/2}$, of x in $B(H)$, where $v \in B(H)$ is a partial isometry with the property* $\ker(v) = \ker(x)$ *(see (Theorem 0.11)). Then $v \in M$ (and clearly $|x| \in M$).*

Proof: Let $u \in M'$ be a unitary. Then:

$$\begin{aligned} x = uxu^* &= uv|x|u^* \\ &= uvu^*|x| && (|x| \in M \text{ and } uy = yu \text{ for all } y \in M) \\ &= uvu^*|x|. \end{aligned}$$

Now uvu^* is a partial isometry with

$$\ker(uvu^*) = u\ker(v) = u\ker(x) = \ker(uxu^*) = \ker(x).$$

Thus, the uniqueness of the polar decomposition yields that $v = uvu^*$ and thus that $vu = uv$. By Lemma 6.13, we get that $v \in M$. ■

7 The Kaplansky density theorem

Motivation 7.1: Let $A \subseteq B(H)$ be a *-subalgebra. Consider the strong closure $B := \mathrm{cl}_{\mathrm{SOT}}(A) \subseteq B(H)$ of A. Then, for each $x \in B$ we find a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that $x_\lambda \to_{\mathrm{SOT}} x$. In general, there is no reason, why $(\|x_\lambda\|)_{\lambda\in\Lambda}$ should be bounded, but can we make a better choice such that $\sup_{\lambda\in\Lambda}\|x_\lambda\| \leq \|x\|$?

Theorem 7.2: *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous and bounded function. Then f is* SOT-continuous*, i. e., if $x_\lambda \to_{\mathrm{SOT}} x$ in $B(H)$ and $x_\lambda = x_\lambda^*$, $x = x^*$, then $f(x_\lambda) \to_{\mathrm{SOT}} f(x)$.*

Proof: ① Consider first the case $f \in C_0(\mathbb{R}) \subset C_b(\mathbb{R})$. Put

$$\mathcal{F}_0 := \{f \in C_0(\mathbb{R}) \mid f \text{ is SOT-continuous}\}.$$

Then $\mathcal{F}_0 \subseteq C_0(\mathbb{R})$ is a subalgebra of $C_0(\mathbb{R})$ that is closed with respect to the sup-norm (Check!). Furthermore, since $x \mapsto x^*$ is continous with respect to SOT on the set of normal operators (note that $x \in B(H)$ is normal if and only if $\|x\xi\| = \|x^*\xi\|$ for all $\xi \in H$), we have that $\mathcal{F}_0$ forms a *-subalgebra of $C_0(\mathbb{R})$. Consider now

$$\begin{aligned} g\colon \mathbb{R} &\longrightarrow \mathbb{C} \\ t &\longmapsto \frac{t}{1+t^2}. \end{aligned}$$

Clearly, $g \in C_0(\mathbb{R})$. *Claim: $g \in \mathcal{F}_0$.*

Proof: Take a net $(x_\lambda)_{\lambda\in\Lambda}$ of selfadjoint operators in $B(H)$ that converges to $x = x^* \in B(H)$ with respect to SOT. Then

$$\begin{aligned} &g(x_\lambda) - g(x) \\ &= x_\lambda(1+x_\lambda^2)^{-1} - x(1+x^2)^{-1} \\ &= (1+x_\lambda^2)^{-1}\left[x_\lambda(1+x^2) - (1-x_\lambda^2)x\right](1+x^2)^{-1} \\ &= (1+x_\lambda^2)^{-1}\left[(x_\lambda - x) + x_\lambda(x-x_\lambda)x\right](1+x^2)^{-1} \end{aligned}$$

so that for each $\xi \in H$ we have:

$$\begin{aligned} &\|g(x_\lambda)\xi - g(x)\xi\| \\ &\leq \|(1+x_\lambda^2)^{-1}(x_\lambda - x)(1+x^2)^{-1}\xi\| + \|(1+x_\lambda^2)^{-1}x_\lambda(x_\lambda - x)x(1+x^2)^{-1}\xi\| \\ &\leq \underbrace{\|(1+x_\lambda^2)^{-1}\|}_{\leq 1}\underbrace{\|(x_\lambda - x)(1+x^2)^{-1}\xi\|}_{\to 0} + \underbrace{\|(1+x_\lambda^2)^{-1}x_\lambda\|}_{\leq 1}\underbrace{\|(x_\lambda - x)x(1+x^2)^{-1}\xi\|}_{\to 0} \end{aligned}$$

because of the properties of the functional calculus and $(1+x^2)^{-1}\xi, x(1+x^2)^{-1}\xi \in H$ are fixed, i. e. $g(x_\lambda) \to_{\mathrm{SOT}} g(x)$. ■

Analogously, $f \in \mathcal{F}_0$ where $f\colon \mathbb{R} \to \mathbb{C}$, $t \mapsto \frac{1}{1+t^2}$. Note that $\{f, g\}$ separates the points of $\mathbb{R}$ and $f(t) > 0$ for all $t \in \mathbb{R}$ and thus, by the Stone-Weierstraß theorem for $C_0(\mathbb{R})$, we get that $\mathcal{F}_0 = C_0(\mathbb{R})$.

② Cosider now the case $f \in C_b(\mathbb{R})$. Put

$$\mathcal{F} := \{f \in C(\mathbb{R}) \mid f \text{ is SOT continuous}\}.$$

Note that for $h_1, h_2 \in \mathcal{F}$, where one h_i is bounded, it holds that $h_1 h_2$ is bounded. Thus in particular: If $h \in \mathcal{F}$ is bounded, then $\mathrm{id} h \in \mathcal{F}$.

Take $h \in C_b(\mathbb{R})$. Then, with f, g constructed like in ①, we have $hf, hg \in \mathcal{F}$. Since $f + \mathrm{id} g = 1$, we may deduce that

$$h = h(f + \mathrm{id} g) = hf + \mathrm{id}(hg) \in \mathcal{F}$$

with the above arguments. ■

Let $S \subseteq B(H)$ be any subset. In the following we will denote with

$$S_{sa} := \{x \in S \mid x^* = x\}$$

the *selfadjoint part of* S.

Theorem 7.3 (Kaplansky density theorem): *Let $A \subseteq B(H)$ be a *-algebra. Consider $B := \mathrm{cl}_{\mathrm{SOT}}(A) \subseteq B(H)$. Then the following statements holds true:*

(i) $\mathrm{cl}_{\mathrm{SOT}}(A_{sa}) = B_{sa}$,

(ii) $\mathrm{cl}_{\mathrm{SOT}}(\{x \in A_{sa} \mid \|x\| \leq 1\}) = \{x \in B_{sa} \mid \|x\| \leq 1\}$,

(iii) $\mathrm{cl}_{\mathrm{SOT}}(\{x \in A \mid \|x\| \leq 1\}) = \{x \in B \mid \|x\| \leq 1\}$.

Proof: (i) Since A_{sa} is convex, we have by Corollary 6.5 that $\mathrm{cl}_{\mathrm{SOT}}(A_{sa}) = \mathrm{cl}_{\mathrm{WOT}}(A_{sa})$. Now, since $x \mapsto x^*$ is weakly continuous, we get that $\mathrm{cl}_{\mathrm{WOT}}(A_{sa}) \subseteq B_{sa}$. Conversely, if $x \in B_{sa} \subseteq B$ is given, then we find a net $(x_\lambda)_{\lambda \in \Lambda}$ in A such that $x_\lambda \to_{\mathrm{SOT}} x$. Then $x_\lambda \to_{\mathrm{WOT}}$ and thus $x_\lambda^* \to_{\mathrm{WOT}} x^* = x$. Hence $A_{sa} \ni \mathrm{Re}(x_\lambda) \to_{\mathrm{WOT}} x$ and thus $x \in \mathrm{cl}_{\mathrm{WOT}}(A_{sa})$. In summary we have $\mathrm{cl}_{\mathrm{SOT}}(A_{sa}) = \mathrm{cl}_{\mathrm{WOT}}(A_{sa}) = B$.

(ii) Take $x \in B_{sa}$ with $\|x\| \leq 1$. Due to (i), we find a net $(x_\lambda)_{\lambda \in \Lambda}$ in A_{sa} with $x_\lambda \to_{\mathrm{SOT}} x$. Consider $f \in C_0(\mathbb{R})$ that is defined by

$$f(t) = \begin{cases} t, & t \in [-1, 1], \\ \frac{1}{t}, & t \notin [-1, 1]. \end{cases}$$

From Theorem 7.2, it follows that $(\mathrm{cl}_{\mathrm{ONT}}(A))_{sa} \ni f(x_\lambda) \to_{\mathrm{SOT}} f(x) = x$, where we used that $\mathrm{Sp}(x) \subseteq [-1, 1]$, and $\|f(x_\lambda)\| \leq 1$ since $\|f\|_\infty \leq 1$. Now, we have: For all $\lambda \in \Lambda$ and all $n \in \mathbb{N}$ there is $y_{n,\lambda} \in A_{sa}$ such that

$$\|y_{\lambda,n}\| \leq 1 \quad \text{and} \quad \|f(x_\lambda) - y_{\lambda,n}\| < \frac{1}{n}.$$

Thus, $(y_{\lambda,n})_{\lambda \in \Lambda, n \in \mathbb{N}}$ is a net in A_{sa} that converges to x in the strong operator topology and satisfies $\|y_{\lambda,n}\| \leq 1$. This shows "$\supseteq$". We leave "$\subseteq$" as an exercise.

(iii) Take $x \in B$ with $\|x\| \leq 1$. Consider

$$y := \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in M_2(B) \subseteq B(H \oplus H).$$

In fact, $y \in M_2(B)_{sa}$. Note that a net $(y_\lambda)_{\lambda \in \Lambda}$ with

$$y_\lambda = \begin{pmatrix} y_\lambda^{11} & y_\lambda^{12} \\ y_\lambda^{21} & y_\lambda^{22} \end{pmatrix} \in M_2(B)$$

is strongly convergent to

$$y = \begin{pmatrix} y^{11} & y^{12} \\ y^{21} & y^{22} \end{pmatrix} \in M_2(B)$$

if and only if $y_\lambda^{ij} \to_{\text{SOT}} y^{ij}$ for all $1 \leq i, j \leq 2$. Thus, $M_2(B) = \text{cl}_{\text{SOT}}(M_2(A))$. Due to (ii), we find a net $(y_\lambda)_{\lambda \in \Lambda}$ in $M_2(A)_{sa}$ such that

$$\begin{pmatrix} y_\lambda^{11} & y_\lambda^{12} \\ y_\lambda^{21} & y_\lambda^{22} \end{pmatrix} = y_\lambda \to_{\text{SOT}} y = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}$$

and $\|y_\lambda\| \leq 1$. Hence $x_\lambda := y_\lambda^{12} \to_{\text{SOT}} x$ and $\|x_\lambda\| \leq 1$. The latter follows from

$$\left\langle y_\lambda \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right\rangle = \langle x_\lambda \xi, \eta \rangle$$

for all $\xi, \eta \in H$. This shows the inclusion "$\supseteq$". Again, we leave "$\subseteq$" as an exercise.
■

Remark 7.4: (i) Note that (ii) in Theorem 7.3 is not a trivial consequence of (iii) as $x \mapsto x^*$ is not continuous with respect to SOT (see (Remark 1.3)).

(ii) One can show that in the situation of Theorem 7.3 also

$$\text{cl}_{\text{SOT}}(\{x \in A \text{ positive} \mid \|x\| \leq 1\}) = \{x \in B \text{ positive} \mid \|x\| \leq 1\}$$

and, if A' is a C^*-algebra:

$$\text{cl}_{\text{SOT}}(\{x \in A' \text{ unitary}\}) = \{x \in B \mid x \text{ unitary}\}.$$

Corollary 7.5: *Let $A \subseteq B(H)$ be a *-algebra. Then, for each $x \in B := \text{cl}_{\text{SOT}}(A)$, we find a net $(x_\lambda)_{\lambda \in \Lambda}$ in A such that*

$$x_\lambda \to_{\text{SOT}} x \quad \textit{and} \quad \sup_{\lambda \in \Lambda} \|x_\lambda\| \leq \|x\|.$$

If x is selfadjoint, then $(x_\lambda)_{\lambda \in \Lambda}$ can be chosen to consist of selfadjoint operators x_λ.

If the underlying Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is separable, then there exists even a sequence $(x_n)_{n \in \mathbb{N}}$ in A with the above properties.

Proof: Exercise! ■

8 Abelian von Neumann algebras

Motivation 8.1: Let A be a commutative unital C^*-algebra. According to the Gelfand-Naimark Theorem (Theorem 10.16 from the Functional Analysis I lecture notes), the Gelfand transformation

$$\chi\colon A \longrightarrow C(\mathrm{Spec}(A)), \qquad x \longmapsto \hat{x}$$

where $\hat{x}(\varphi) := \varphi(x)$, gives an isometric *-isomorphism, i. e., A is of the form $C(K)$ for some compact Hausdorff space K. Therefore, the theory of C^*-algebras can be seen as a kind of "non-commutative topology".

Analogously, von Neumann algebra theory is considered as a kind of "non-commutative measure theory"; in order to understand this point of view, we study here abelian (i. e., commutative) von Neumann algebras.

Remark 8.2: Since every abelian von Neumann algebra is particular a commutative C^*-algebra, it is isomorphic to $C(K)$ for some compact Hausdorff space K; this space, however, is *extremally disconnected* (i. e., the closure of every open set is again open). This highlights that von Neumann algebras are rather exceptional among C^*-algebras.

Definition 8.3: Let $M \subseteq B(H)$ be a von Neumann algebra and let $0 \neq \xi \in H$ be given. We say that

(i) ξ is *cyclic for* M, if $M\xi = \{x\xi \mid x \in M\}$ is dense in H,

(ii) ξ is *separating for* M, if $x\xi \neq 0$ for all $0 \neq x \in M$.

Theorem 8.4: *Let $M \subseteq B(H)$ be a von Neumann algebra and let $0 \neq \xi \in H$ be given. Then ξ is cyclic for M if and only if ξ is separating for M'.*

Proof: "$\Rightarrow$": Let ξ be cyclic for M. Suppose that $x\xi = 0$ for some $x \in M'$. Then for all $y \in M$ we have

$$xy\xi = yx\xi = y0 = 0.$$

Since $M\xi$ is dense in H, we infer that even $x\eta = 0$ for all $\eta \in H$, hence $x = 0$. Thus, ξ is separation for M'.

"$\Leftarrow$": Let ξ be separating for M'. Assume that $M\xi$ was not dense in H. Then the orthogonal projection p onto $\mathrm{cl}(M\xi)^\perp$ was non-zero but we had

$$0 = \langle p\xi, \xi\rangle = \langle p\xi, p\xi\rangle = \|p\xi\|^2, \qquad\qquad (p\xi \in M\xi^\perp)$$

i. e., $p\xi = 0$ and $p \in M'$ (which follows from step ① in the proof of Theorem 6.10) which was a contradiction. ■

Corollary 8.5: *If $M \subseteq B(H)$ is an abelian von Neumann algebra and $\xi \in H$ cyclic for M, then ξ is also separating for M.*

Proof: If ξ is cyclic for M, then ξ is separating for M' due to Theorem 8.4. Now, because being separating passes to von Neumann subalgebras, the assertion follows since $M \subseteq M'$. ■

Definition 8.6: A von Neumann algebra $M \subseteq B(H)$ is said to be *separable*, if the underlying Hilbert space H is separable (see Remark 0.3 (vii)).

Theorem 8.7: *Let $M \subseteq B(H)$ be a separable abelian von Neumann algebra, then there exists a separating vector $0 \neq \xi \in H$ for M.*

Proof: By Zorns Lemma, there exists a maximal family of non-zero unit vectors $(\xi_\lambda)_{\lambda\in\Lambda}$ such that $M\xi_\lambda \perp M\xi_\mu$ for all $\lambda, \mu \in \Lambda$ where $\lambda \neq \mu$. Since H is separable, that family is countable, say $(\xi_n)_{n=1}^R$, $R \in \mathbb{N} \cup \{\infty\}$. Its maximality implies that

$$H_0 := \{x\xi_n \mid n \in \mathbb{N}, n \leq R, x \in M\} \subseteq H$$

is dense in H. Denote by p_n the orthogonal projection onto $\mathrm{cl}(M\xi_n)$. Then (see step ① in the proof of Theorem 6.10) we have $p_n \in M'$. Put

$$\xi := \sum_{n=1}^{R} \frac{1}{2^n}\xi_n \in H \setminus \{0\}$$

(note that $(\xi_n)_{n=1}^R$ is orthonormal). Then ξ is separating for M: Take $x \in M$ such that $x\xi = 0$. Then for all $n \in \mathbb{N}, n \leq R$ we have

$$0 = p_n(x\xi) = xp_n\xi = x\frac{1}{2^n}\xi^n = \frac{1}{2^n}x\xi_n,$$

so for all $n \in \mathbb{N}$ we have $x\xi_n = 0$, i.e., for all $y \in M$ and $n \in \mathbb{N}$, $n \leq R$ it holds $0 = y(x\xi_n) = x(y\xi_n)$. Thus for all $\eta \in H_0$ it holds $x\eta = 0$ and as H_0 is dense in H, we get $x = 0$. ■

Definition 8.8: An abelian von Neumann algebra $M \subseteq B(H)$ is called *maximal*, if $M \subseteq N \subseteq B(H)$ for an abelian von Neumann algebra $N \subseteq B(H)$ implies that $M = N$.

Lemma 8.9: *An abelian von Neumann algebra $M \subseteq B(H)$ is maximal if and only if $M' = M$ holds.*

Proof: Exercise. ■

Corollary 8.10: *Let $M \subseteq B(H)$ be a separable maximal abelian von Neumann algebra. Then there exists a cyclic vector $0 \neq \xi \in H$ for M.*

Proof: By Theorem 8.7, there is a separating vector $0 \neq \xi \in H$ for M. Note that $M = M'$ due to Lemma 8.9. Thus, Theorem 8.4 implies that ξ is also cyclic for M. ■

Example 8.11: Let K be a compact Hausdorff space and let μ be a finite Borel measure on K. For each function $f \in L^\infty(K, \mu)$, we define $M_f \in B(L^2(K, \mu))$ by $M_f g := fg$ for all $g \in L^2(K, \mu)$. We then have that $\|M_f\| = \|f\|_{L^\infty(K,\mu)}$ for all $f \in L^\infty(K, \mu)$. Consider the C^*-algebra

$$A := \{M_f \mid f \in C(K)\}.$$

Then we have the statements:

(i) $M := A' = \{M_f \mid f \in L^\infty(K, \mu)\}$, which is a von Neumann algebra acting on $L^2(K, \mu)$.

(ii) $A'' = A'$ and thus $M' = M$, i. e., M is maximal abelian.

(iii) The constant function 1 is cyclic and separating for M.

Theorem 8.12: *Let $M \subseteq B(H)$ be an abelian von Neumann algebra and suppose that there is a cyclic vector $0 \neq \xi \in H$ for M. Then, for any* SOT*-dense unital C^*-subalgebra $A \subseteq M$ there exists a finite Radon measure μ on $K :=$* Spec(A) *with* supp$(\mu) = K$ *and an unitary $U: L^2(K, \mu) \to H$ such that*

$$U^*MU = \{M_f \mid f \in L^\infty(K, \mu)\} (\cong L^\infty(K, \mu)) \subseteq B(L^2(K, \mu)).$$

Remark 8.13: A finite Borel measure μ on a Hausdorff space X is a finite *Radon measure*, if it is *inner regular*, i. e., if for each Borel set $B \in \mathfrak{B}(X)$ we have

$$\mu(B) = \sup_{K \subseteq B \text{ compact}} \mu(K).$$

Those Radon measures are automatically *outer regular*, i. e., we have for each $B \in \mathfrak{B}(X)$ that

$$\mu(B) = \inf_{B \subseteq U \text{ open}} \mu(U).$$

We define the *support* of a finite Radon measure by

$$\operatorname{supp}(\mu) := X \setminus V,$$

where $V := \bigcup\{U \subseteq X \text{ open} \mid \mu(U) = 0\}$.

Proof: ① By the Riesz representation theorem, we find a Radon measure μ on the compact Hausdorff space $K := \operatorname{Spec}(A)$ such that

$$\langle \chi^{-1}(f)\xi, \xi\rangle = \int_K f(x)\, d\mu(x)$$

for all $f \in C(K)$, where $\chi^{-1}: C(K) \to A$ is the inverse Gelfand transform, which is positive. Define

$$U_0: C(K) \longrightarrow H, \qquad f \longmapsto \chi^{-1}(f)\xi.$$

Then, for all $f \in C(K)$, we have

$$\|U_0 f\|^2 = \langle \chi^{-1}(f)\xi, \chi^{-1}(f)\xi\rangle = \langle \chi^{-1}(|f|^2)\xi, \xi\rangle = \int_K |f(x)|^2 \, d\mu(x) = \|f\|^2_{L^2(K,\mu)}$$

so that U_0 extends to an isometry $U\colon L^2(K,\mu) \to H$.

② *Claim:* $U(L^2(K,\mu)) = H$, i. e., U is a unitary. Note that $A\xi = U_0(C(K)) \subseteq U(L^2(K,\mu))$ and $\mathrm{cl}(A\xi) = \mathrm{cl}(M\xi) = H$ (as $A \subseteq M$ is SOT-dense and ξ is cyclic), thus $U(L^2(K,\mu)) = H$.

③ *Claim:* $\mathrm{supp}(\mu) = K$. Otherwise, there would be some $f \in C(K)$ such that

$$0 = \int_K |f(x)|^2 \, d\mu(x) = \|\chi^{-1}(f)\xi\|^2 \quad \Longrightarrow \quad \chi^{-1}(f)\xi = 0$$

which contradicts the fact that ξ is also separating for M due to Corollary 8.5.

④ *Claim:* For all $f \in C(K) \subseteq L^\infty$ it holds: $U^*\chi^{-1}(f)U = M_f$. Take any $g \in C(K) \subseteq L^2(K,\mu)$. Then

$$U^*\chi^{-1}(f)Ug = U^*\chi^{-1}(f)\chi^{-1}(g)\xi = U^*\chi^{-1}(fg)\xi = fg = M_f g.$$

Since $C(K)$ is dense in $L^2(K,\mu)$, we conclude that $U^*\chi^{-1}(f)U = M_f$. Thus, we see that $U^*AU = \{M_f \mid f \in C(K)\}$.

⑤ *Claim:* $U^*MU = \{M_f \mid f \in L^\infty(K,\mu)\}$. Consider the *-isomorphism

$$\mathrm{Ad}(U)\colon B(L^2(K,\mu)) \longrightarrow B(H), \qquad x \longmapsto UxU^*$$

with inverse given by $\mathrm{Ad}(U^*)\colon B(H) \to B(L^2(K,\mu))$. Observe that $\mathrm{Ad}(U)$ and $\mathrm{Ad}(U^*) = (\mathrm{Ad}(U))^{-1}$ are obviously SOT-continuous. Thus, since $\mathrm{cl}_{\mathrm{SOT}}(A) = M$ and also $\mathrm{cl}_{\mathrm{SOT}}(\{M_f \mid f \in C(K)\}) = \{M_f \mid f \in L^\infty(K,\mu)\}$ (where this equality is shown in Example 8.11) we infer from $\mathrm{Ad}(U^*)(A) = \{M_f \mid f \in C(K)\}$ (which holds by ④), that

$$\mathrm{Ad}(U^*)(M) = \{M_f \mid f \in L^\infty(K,\mu)\}$$

as desired. ■

Remark 8.14: In Corollary 8.10, we have seen that if a von Neumann algebra is maximal abelian then it has a cyclic vector $\xi \in H$ — under the hypothesis that H is separable. Theorem 8.12 (if combined with Example 8.11) says that the converse is also true — even without the separability hypothesis.

Theorem 8.15: *Let $M \subseteq B(H)$ be a separable abelian von Neumann algebra. Then there exists a compact Hausdorff space K with a finite Radon measure μ such that M and $L^\infty(K,\mu)$ are *-isomorphic.*

Proof: By Theorem 8.7 there exists a separating vector ξ for M. Consider the sub Hilbert space $H_0 := \mathrm{cl}(M\xi) \subseteq H$. Then

$$\varphi\colon M \longrightarrow B(H_0), \qquad x \longmapsto x|_{H_0}$$

is a well-defined (since x maps H_0 to itself) and injective (since $\xi_0 \in H_0$ is separating for M) *-homomorphism that is unital. Thus φ is isometric (Proposition 4.10) so that $\varphi(M)$ is a C^*-algebra on H_0. Now consider $M_0 = \mathrm{cl}_{\mathrm{SOT}}(\varphi(M)) \subseteq B(H_0)$. Then ξ is cyclic for M_0, because $M_0\xi \supseteq \varphi(M)\xi = M\xi$ and $\mathrm{cl}(M\xi) = H_0$, hence we may apply Theorem 8.12 which gives $U^*M_0U = \{M_f \mid f \in L^\infty(K,\mu)\}$ and thus a *-isomorphism $\psi_0\colon M_0 \to L^\infty(K,\mu)$ with

$$\psi_0^{-1}\colon L^\infty(K,\mu) \longrightarrow M_0, \qquad f \longmapsto UM_fU^*.$$

We are done, if we can prove that infact $M_0 = \varphi(M)$. To see this, take $y \in M_0$. By Corollary 7.5, we find a net $(y_\lambda)_{\lambda\in\Lambda}$ in $\varphi(M)$ such that $y_\lambda \to y$ in the strong operator topology and $\|y_\lambda\| \le \|y\|$ for all $\lambda \in \Lambda$. Thus, we find a net $(x_\lambda)_{\lambda\in\Lambda}$ in M such that $y_\lambda = \varphi(x_\lambda)$ and

$$\|x_\lambda\| \le \|\varphi(x_\lambda)\| = \|y_\lambda\| \le \|y\|$$

for all $\lambda \in \Lambda$. Since $\{x \in M \mid \|x\| \le r\}$ is compact with respect to the weak operator topology for all $r > 0$, there is a subnet $(x_{\lambda(\gamma)})_{\gamma\in\Gamma}$ of $(x_\lambda)_{\lambda\in\Lambda}$ such that $x_{\lambda(\gamma)} \to_{\mathrm{WOT}} x$ for some $x \in M$ ($\|x\| \le \|y\|$). Thus, since φ is WOT-continuous, $\varphi(x_{\lambda(\gamma)}) \to_{\mathrm{WOT}} \varphi(x)$ and $\varphi(x_{\lambda(\gamma)}) \to_{\mathrm{WOT}} y$ (since $\varphi(x_\lambda) \to_{\mathrm{SOT}} y$).

Thus by the uniqueness of the limit $y = \varphi(x) \in \varphi(M)$ which is what we wanted to show. ■

Remark 8.16: (i) We have used $\{x \in B(H) \mid \|x\| \le r\} =: B(H)_r$ and hence $\{x \in M \mid \|x\| \le r\} = M \cap B(H)_r$ are compact with respect to the weak operator topology for all $r > 0$. This can be seen as follows: Consider the map

$$\iota\colon B(H)_r \longrightarrow \prod_{\zeta,\eta\in H} \{z \in \mathbb{C} \mid |z| < r\|\zeta\|\|\eta\|\} =: K, \qquad x \longmapsto (\langle x\zeta,\eta\rangle)_{\zeta,\eta\in H}.$$

It is clearly injecitve and moreover continuous if $B(H)_r$ is endowed with the weak operator topology and K with the product topology; in fact, one can show that $\iota\colon B(H)_r \to \iota(B(H)_r)$ is a homeomorphism. By Tychonoff's theorem, K is compact; thus it suffices to verify that $\iota(B(H)_r)$ is closed. If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in $B(H)_r$ for which $(\iota(x_\lambda))_{\lambda\in\Lambda}$ converges to some point $z = (z_{\zeta,\eta})_{\zeta,\eta\in H}$, one finds a linear map $x\colon H \to H$ such that $\langle x\zeta,\eta\rangle = z_{\zeta,\eta}$ for all ζ,η in H. Thus, it follows $\|x\| \le r$ since

$$|\langle x\zeta,\eta\rangle| = |z_{\zeta,\eta}| \le r\|\zeta\|\|\eta\|.$$

Hence $x \in B(H)_r$ with $\iota(x) = z$.

(ii) If H is separable and $x \in B(H)$ is normal, we may apply Theorem 8.12 to $A = C^*(x, 1)$ and $M = \mathrm{vN}(x)$, which yields a *-isomorphism $L^\infty(\mathrm{Spec}(A), \mu) \to M$ for some finite Radon measure μ. Since $\mathrm{Spec}(A) \cong \mathrm{Sp}(x)$ due to Lemma 10.22 (from the Functional Analysis I lecture notes), we obtain a *-ismorphism

$$\tilde{\Phi}\colon L^\infty(\mathrm{Sp}(x), \nu) \longrightarrow M$$

with the induced measure ν on $\mathrm{Sp}(x)$, that agrees with the measurable functional calculus; see (Remark 6.12) (iv).

(iii) Let $M \subseteq B(H)$ be a separable abelian von Neumann algebra. One can show that there exists a selfadjoint element $x = x^* \in M$ such that $M = \mathrm{vN}(x)$.

(iv) From (iii), one can deduce that for each separable abelian von Neumann algebra $M \subseteq B(H)$ a countable (possibly empty) set I exists such that M is *-isomorphic to either $\ell^\infty(I)$ or $L^\infty([0,1], \lambda^1)$[1].

(v) Theorem 8.15 stays valid of the separability condition is removed. In that case one has to decompose H as $H = \bigoplus_{\lambda \in \Lambda} \mathrm{cl}(M\xi_\lambda)$ for an orthonormal family $(\xi_\lambda)_{\lambda \in \Lambda}$ like in the proof of Theorem 8.7. Let p_λ be the orthogonal projection onto $H_\lambda := \mathrm{cl}(M\xi_\lambda)$. We know $p_\lambda \in M'$; one can show that $Mp_\lambda \subseteq B(H_\lambda)$ is an abelian von Neumann algebra with cyclic vector ξ_λ. Thus, by Theorem 8.15, Mp_λ is *-isomorphic to $L^\infty(K_\lambda, \mu_\lambda)$ for some *probability space* (K_λ, μ_λ). Then M is *-isomorphic to

$$\bigoplus_{\lambda \in \Lambda} L^\infty(K_\lambda, \mu_\lambda) \cong L^\infty\Big(\prod_{\lambda \in \Lambda} K_\lambda, \prod_{\lambda \in \Lambda} \mu_\lambda\Big)$$

[1] Here, λ^1 denotes the one-dimensional Lebesgue measure on $\mathfrak{B}([0,1])$.

9 Projections, factors and their type classification

Motivation 9.1: We have already seen (see Remark 6.12 (iv)), that von Neumann algebras contain typically "many" projections, i. e., operators e that satisfy $e^2 = e = e^*$ (see Definition 0.10 (vi)). Thus, by studying projections, we gain some deeper understanding of the structure of von Neumann algebras.

Definition 9.2: Let $M \subseteq B(H)$ be a von Neumann algebra and let $e, f \in M$ be two projections. We say that

(i) e and f are *equivalent* ($e \sim f$), if there exists a partial isometry $u \in M$ (see Definition 0.10 (vii)), such that

$$u^*u = e \quad \text{and} \quad uu^* = f.$$

This is also called *Murray-von-Neumann equivalence.*

(ii) e is subordinate to f ($e \precsim f$), if there exists a projection $g \in M$ such that $e \sim g \leq f$.

Remark 9.3: (i) The partial isometry that provides the equivalence must belong to M; "M knows that the projections are equivalent". Thus, both $\sim$ and $\precsim$ are relative to M. In fact, we can have $f \sim e$ in $B(H)$ but $f \nsim e$ in M (namely if $u \notin M$).

(ii) For $M = B(H)$, we have the following:

$$e \sim f \Leftrightarrow \dim eH = \dim fH$$
$$e \precsim f \Leftrightarrow \dim eH \leq \dim fH$$

(iii) $g \leq f$ means $f - g$ is positive. Note that

$$f - g \geq 0 \Leftrightarrow gf = g \Leftrightarrow fg = g \Leftrightarrow gH \subseteq fH.$$

(iv) "$\sim$" is an *equivalence relation*, i. e.,

- $e \sim e$ (reflexivity)
- $e \sim f \Leftrightarrow f \sim e$ (symmetry)
- $(e \sim f \wedge f \sim g) \Rightarrow e \sim g$ (transitivity)

"$\precsim$" is easily seen to be a *preorder*, i. e.,

- $e \precsim e$,
- $(e \precsim f \wedge f \precsim g) \Rightarrow e \precsim g$.

We will see that "$\precsim$" is in fact a *partial order*, i. e., it satsifies also

- $(e \precsim f \wedge f \precsim e) \Rightarrow e \sim f$. (antisymmetry)

Under a further condition on M, "$\precsim$" is even a *total order* on

$$\mathcal{P}(M) := \{p \in M \mid p \text{ projection}\},$$

i. e., any two $e, f \in \mathcal{P}(M)$ are *comparable* in the sense that $e \precsim f$ or $f \precsim e$ must hold.

Theorem 9.4: (i) *Let $(e_i)_{i\in I}$ be a family of mutually orthogonal projections in $B(H)$ (i. e., we have $e_ie_j = 0$ for all $i, j \in I$ with $i \neq j$). Then $\sum_{i\in I} e_i$ converges in the strong operator topology to a projection e.*

(ii) *Let $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ be two families of mutually orthogonal projections in a von Neumann algebra $M \subseteq B(H)$. Then:*

- $e_i \sim f_i \,\forall\, i \in I \Rightarrow \sum_{i\in I} e_i \sim \sum_{i\in I} f_i$,
- $e_i \precsim f_i \,\forall\, i \in I \Rightarrow \sum_{i\in I} e_i \precsim \sum_{i\in I} f_i$.

Proof: (i) Put $K_i := e_iH$ and $K := \bigoplus_{i\in I} K_i \subseteq H$. Let e be the projection onto K. We want to show that

$$\Big(\sum_{i\in F} e_i\Big)_{F\subseteq_{\text{fin}} I} \to_{\text{SOT}} e.$$

① For $\xi \in K^\perp$, we have $\xi \in K_i^\perp$ for all $i \in I$. Thus, $\sum_{i\in F} e_i\xi = 0$ for all $F \subseteq_{\text{fin}} I$, thus $(\sum_{i\in F} e_i\xi)_{F\subseteq_{\text{fin}} I}$ converges to $0 = e\xi$ in H.

② For $\xi \in K$, we find by definition of K a summable family $(\xi_i)_{i\in I}$ with $\xi_i \in K_i$ for $i \in I$, that has the sum ξ, i. e., by Remark 5.23 (ii) from the Functional Analysis I lecture notes we get

$$\Big(\sum_{i\in F} \xi_i\Big)_{F\subseteq_{\text{fin}} I} = \Big(\sum_{i\in F} e_i\xi\Big)_{F\subseteq_{\text{fin}} I} \qquad \text{(as } \xi_i = e_i\xi\text{)}$$

converges to $\xi = e\xi$ in H.

Thus, for each $\xi \in K \oplus K^\perp = H$, the net $(\sum_{i\in F} e_i\xi)_{F\subseteq_{\text{fin}} I}$ converges to $e\xi$ in H.

(ii) For each $i \in I$, we find a partial isometry $u_i \in M$ such that $u_i^*u_i = e_i$ and $u_iu_i^* = f_i$. Define $u \in B(H)$ by

$$u|_{K_i} := u_i|_{K_i} \quad \text{and} \quad u|_{K^\perp} := 0.$$

Then u is a partial isometry with $u^*u = \sum_{i\in I} e_i$ and $uu^* := \sum_{i\in I} f_i$. Moreover, for each element $y \in M'$, we have

$$yu|_{K_i} = yu_i|_{K_i} = u_iy|_{K_i} = u_iy|_{K_i} = uy|_{K_i} \qquad \text{(as } y \in M' \text{ and } ye_i = e_i\text{)}$$

for all $i \in I$, thus $yu = uy$; this means $u \in M'' = M$.

For the second part: For each $i \in I$ we find a projection $g_i \in M$ such that $e_i \sim g_i \leq f_i$. Then (via the previous result):

$$\sum_{i \in I} e_i \sim \sum_{i \in I} g_i \leq \sum_{i \in I} f_i,$$

i. e., $\sum_{i \in I} \precsim \sum_{i \in I} f_i$. ■

Theorem 9.5 (Cantor-Bernstein-Schröder theorem for projections): *Let $M \subseteq B(H)$ be a von Neumann algebra and let $e, f \in M$ be two projections. Then:*

$$e \sim f \iff e \precsim f \text{ and } f \precsim e.$$

Thus, $\precsim$ is a partial order on $\mathcal{P}(M)$ (or, more precisely, on $\mathcal{P}(M)/\sim$).

Proof: "⇒" is obvious, since $e \sim f \leq f$ and $f \sim e \leq e$.
Now for "⇐": By assumption, we have

$$e \precsim f \Rightarrow \exists u \in M \text{ partial isometry} : e = u^*u \text{ and } uu^* \leq f$$
$$f \precsim e \Rightarrow \exists v \in M \text{ partial isometry} : f = v^*v \text{ and } vv^* \leq e$$

Note that $u|_{eH} \colon eH \to fH$ and $v|_{fH} \colon fH \to eH$ are both isometries (see Definition 0.10 (vii)). Define:

- $e_0 := e - v^*v \leq e$, $f_0 := ue_0e^* \leq uu^* \leq f$ — note that
$$f_0^2 = ue_0u^*ue_0u^* = ue_0ee_0u^* = ue_0e^* = f_0.$$
- $e_n := vf_{n-1}v^* \leq vv^* \leq e$, $f_n := ue_nu^* \leq uu^* \leq f$.
- $e_\infty := e - \sum_{n \in \mathbb{N}} e_n$, $f_\infty := f - \sum_{n \in \mathbb{N}} f_n$

Note that $(e_n)_{n \in \mathbb{N}}$ are mutually orthogonal projections in M and $(f_n)_{n \in \mathbb{N}}$ are mutually orthogonal projections in M. Furhtermore note that the convergence of the series $\sum_{n \in \mathbb{N}} e_n$, $\sum_{n \in \mathbb{N}} f_n$ is guaranteed by Theorem 9.4 (i).

① *Claim: $e_n \sim f_n$ for all $n \in \mathbb{N}$.*
Indeed:
- $(ue_n)^*(ue_n) = e_nu^*ue_n = e_nee_n = e_n^2 = e_n$,
- $(ue_n)(ue_n)^* = ue_ne_nu^* = ue_nu^* = f_n$

where ue_n is a partial isometry as $ue_n|_{e_nH} \colon e_nH \to eH$ is isometric and $ue_n|_{(e_nH)^\perp} = 0$ (since $(e_nH)^\perp = (1 - e_n)H$) that belongs to M.

② *Claim: $e_\infty \sim f_\infty$.*
Consider $v_N = v^*(e - \sum_{n=0}^N e_n)$. Then

$$v_Nv_N^* = v^*\Big(e - \sum_{n=0}^N e_n\Big)v = v^*(e - e_0)v - \sum_{n=1}^N v^*e_nv$$

$$= v^*vv^*v - \sum_{n=1}^{N}(v^*v)f_{n-1}(v^*v) = f - \sum_{n=0}^{N-1} f_n$$

and

$$v_N^*v_N = \Big(e - \sum_{n=0}^{N} e_n\Big)vv^*\Big(e - \sum_{n=0}^{N} e_n\Big) = e - \sum_{n=0}^{N} e_n,$$

since $e - \sum_{n=0}^{N} e_n = (e - e_0) - \sum_{n=1}^{N} e_n \leq e - e_0 = vv^*$. Thus, as $N \to \infty$, $v_N \to_{\mathrm{SOT}} v_\infty := v^*e_\infty$ (Theorem 9.4 (i)) and

$$v_\infty^* v \;{}_{\mathrm{WOT}}\!\leftarrow v_N^*v_N = e - \sum_{n=0}^{N} e_n \to_{\mathrm{SOT}} e_\infty \qquad \text{as } N \to \infty$$

thus $v_\infty^*v_\infty = e_\infty$ and analogously $v_\infty v_\infty^* = f_\infty$. Thus, by Theorem 9.4 (ii) it holds

$$e = e_\infty + \sum_{n\in\mathbb{N}} e_n \sim f_\infty + \sum_{n\in\mathbb{N}} f_n = f.$$

■

Theorem 9.6: *Let $M \subseteq B(H)$ be a von Neumann algebra. For $x \in M$, we denote by* $\mathrm{supp}(x)$ *the orthogonal projection in $B(H)$ onto* $\ker(x)^\perp = \mathrm{cl}(\mathrm{im}(x^*))$. *Then* $\mathrm{supp}(x) \in M$ *and moreover* $\mathrm{supp}(x) \sim \mathrm{supp}(x^*)$.

Proof: Consider the polar decomposition $x = v|x|$ of x from Corollary 6.14. Then $v \in M$ is a partial isometry and satisfies (see Theorem 0.11) $v^*v =$ projection onto $\ker(x)^\perp = \mathrm{supp}(x)$ and $vv^* =$ projection onto $\mathrm{cl}(\mathrm{im}(x^*)) = \mathrm{supp}(x^*)$. Thus, $\mathrm{supp}(x), \mathrm{supp}(x^*) \in M$ and $\mathrm{supp}(x) \sim \mathrm{supp}(x^*)$. ■

Definition 9.7: Let $M \subseteq B(H)$ be a von Neumann algebra.

(i) We call $Z(M) := M \cap M'$ the *center of* M,

(ii) If $Z(M) = \mathbb{C}1$, then M is called a *factor*.

Remark 9.8: Every separable von Neumann algebra $M \subseteq B(H)$ can be decomposed as a direct integral

$$M \cong \int_X^{\oplus} M_x \, d\mu(x)$$

over some *standard measure space* (X, μ), corresponding to the decomposition

$$H \cong \int_X^{\oplus} H_x \, d\mu(x)$$

of the underlying Hilbert space H by a *measurable field of Hilbert spaces* $(H_x)_{x\in X}$, where $M_x \subseteq B(H_x)$ is a factor for μ-almost all $x \in X$.

In this sense, factors are seen as the *building blocks* of von Neumann algebras. That theory, however, is technically extensive and we do not go into details here.

Lemma 9.9: *Let $M \subseteq B(H)$ be a factor. Then, for any two non-zero projections $e, f \in M$, there are non-zero projections $e_1, f_1 \in M$ such that $e_1 \leq e$, $f_1 \leq f$ and $e_1 \sim f_1$.*

Proof: ① *Claim: $fMe \neq \{0\}$.*

Consider $K := \mathrm{cl}(MeH) \subseteq H$, which is invariant under M. Thus, the projection p onto K is in M'. Furthermore, we have that K is invariant under M', i. e., for all $x \in M'$ it holds $xK \subseteq K$ and thus $px = xp$ (since also $xK^{\perp} \subseteq K^{\perp}$). Therefore, $p \in M'' = M$. As $p \in M'$ and $p \in M'' = M$, we have $p \in M \cap M' = Z(M) = \mathbb{C}1$, it must hold $p = 0$ or $p = 1$. Since $Kneq\{0\}$ we infer $p = 1$. Thus, $\mathrm{cl}(MeH) = K = H$. Now since $f \neq 0$, we conclude $f\mathrm{cl}(MeH) \neq \{0\}$, i. e., $fMe \neq \{0\}$.

② By ①, we find $z \in M$ such that $x := fze \neq 0$. We have: $fx = x$ and $xe = x$ (i. e., $e^*x = x^*$). From $fx = x$ we deduce $f \cdot \mathrm{supp}(x^*) = \mathrm{supp}(x^*)$ and from $xe = x$ we conclude $e \cdot \mathrm{supp}(x) = \mathrm{supp}(x)$. Thus, by Remark 9.3

$$f_1 := \mathrm{supp}(x^*) \leq f \quad \text{and} \quad e_1 := \mathrm{supp}(x) \leq e$$

and by Theorem 9.6, we also have that $e_1 \sim f_1$. ∎

Theorem 9.10: *Let $M \subseteq B(H)$ be a factor. If $e, f \in M$ are projections, then $e \precsim f$ or $f \precsim e$ holds. So, any two projections in $\mathcal{P}(M)$ are comparable.*

Proof: Consider families $(e_i, f_i)_{i \in I}$ such that

- $(e_i)_{i \in I}$ are mutually orthogonal projections in M,
- $(f_i)_{i \in I}$ are mutually orthogonal projections in M,
- For all $i \in I$: $e_i \leq e$, $f_i \leq f$ and $e_i \sim f_i$

They are partially ordered by inclusion; thus, by Zorn's Lemma, there is a maximal family $(e_i f_i)_{i \in I}$. Then:

$$e' := \sum_{i \in I} e_i \leq e \quad \text{and} \quad f' := \sum_{i \in I} f_i \leq f$$

and $e' \sim f'$ by Theorem 9.4.

Claim: $e' = e$ or $f' = f$.

Indeed, if both $e - e' \neq 0$ and $f - f' \neq 0$, we find by Lemma 9.9 non-zero projections $g \leq e - e'$ and $h \leq f - f'$ such that $g \sim h$. Hence, we can enlarge $(e_i f_i)_{i \in I}$ by (g, h) which contradictis the maximality of $(e_i f_i)_{i \in I}$. Thus "$e = e' \sim f' \leq f \Rightarrow e \precsim f$" or "$f = f' \sim e' \leq e \Rightarrow f \precsim e$". ∎

Definition 9.11: Let $M \subseteq B(H)$ be a von Neumann algebra.

(i) A projection $0 \neq e \in M$ is called *minimal*, if it holds "$(f \in \mathcal{P}(M), f \leq e) \Rightarrow (f = 0 \text{ or } f = e)$".

(ii) A projection $e \in M$ is called *finite*, of "$(f \in \mathcal{P}(M), e \sim f \leq e) \Rightarrow f = e$".

Remark 9.12: (i) Consider $M = B(H)$. Then e is minimal if and only if $\dim eH = 1$ and e is finite if and only if $\dim eH < \infty$.

(ii) In general e is minimal if and only if $eMe = \mathbb{C}e$ and if e is minimal, then e is finite — the converse is false however.

Indeed: If e is minimal, then $e \sim f \leq e$ implies $f = 0$ or $f = e$, where $f = 0$ is not possible as $f \sim e \neq 0$.

Definition 9.13 (Murray-von-Neumann, $\sim$ 1930): Let $M \subseteq B(H)$ be a factor. We say the M is of

(i) *type I*, if M contains a minimal projection.

(ii) *type II*, if M contains no minimal projection but a finite projection.

(iii) *type III*, if M contains no finite projection.

10 Type I factors and tensor products

Factors of type I are isomorphic to $B(H)$ for some suitable Hilbert space H; at the same time, their commutants are also of this particular form. This result is the goal of this chapter, for which tensor products of von Neumann algebras are needed.

Remark 10.1: (i) Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be (complex) Hilbert spaces. On their algebraic tensor product (over $\mathbb{C}$)

$$H_1 \otimes H_2 = \Big\{ \sum_{i=1}^{n} \xi_i \otimes \eta_i : n \in \mathbb{N}, \xi_i \in H, \eta_i \in H \Big\}$$

we may introduce an inner product $\langle\cdot,\cdot\rangle$, which is uniquely determined

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle \xi_1, \xi_2 \rangle_2 \langle \eta_1, \eta_2 \rangle_2$$

for all $\xi_1, \xi_2 \in H_1; \eta_1, \eta_2 \in H_2$. This yields a pre-Hilbert space, whose completion will be denoted by $H_1 \mathbin{\hat{\otimes}} H_2$ and is called the *Hilbert space tensor product of H_1 and H_2*.

(ii) If $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$ are orthonormal bases of H_1 and H_2 respectively, then $(\xi_i \otimes \eta_j)_{(i,j)\in I\times J}$ is an orthonormal basis of $H_1 \mathbin{\hat{\otimes}} H_2$. Thus

$$H_1 \mathbin{\hat{\otimes}} H_2 \cong \bigoplus_{i\in I} \underbrace{\xi_i \otimes H_2}_{\cong H_2}.$$

(iii) If $x \in B(H_1)$ and $y \in B(H_2)$ are given, then

$$(x \otimes y)(\xi \otimes \eta) := (x\xi) \otimes (y\eta) \qquad \text{for all } \xi \in H_1, \eta \in H_2$$

defines a linear operator $x \otimes y \colon H_1 \otimes H_2 \to H_1 \otimes H_2$ that extends uniquely to an operator $x \mathbin{\hat{\otimes}} y \in B(H_1 \mathbin{\hat{\otimes}} H_2)$ with

$$\|x \mathbin{\hat{\otimes}} y\| = \|x\|\|y\|.$$

(iv) Let $M_1 \subseteq B(H_1)$ and $M_2 \subseteq B(H_2)$ be von Neumann algebras. We define their *(von Neumann algebra) tensor product* by

$$M_1 \mathbin{\overline{\otimes}} M_2 = \mathrm{cl}_{\mathrm{WOT}}(\langle\{x \mathbin{\hat{\otimes}} y \mid x \in M_1, y \in M_2\}\rangle) \subseteq B(H_1 \mathbin{\hat{\otimes}} H_2).$$

It is a deep theorem (in fact, it was one of the first applications of the Tomita-Takesaki theory) that

$$(M_1 \mathbin{\overline{\otimes}} M_2)' = M_1' \mathbin{\overline{\otimes}} M_2'.$$

Furthermore one can show that if both M_1 and M_2 are factors, then also $M_1 \mathbin{\overline{\otimes}} M_2$ is a factor.

Lemma 10.2: *Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be complex Hilbert spaces. Let $M \subseteq B(H_2)$ be a von Neumann algebra. Then*

(i) $\iota\colon B(H_2) \to B(H_1 \hat{\otimes} H_2)$, $y \mapsto \mathrm{id}_{H_1} \hat{\otimes} y$ *is* WOT-*continuous and we have*
$$\iota(M) = \iota(M)'' = (\mathbb{C}\mathrm{id}_{H_1}) \hat{\otimes} M,$$
i. e., $\mathrm{id}_{H_1} \otimes M = \iota(M) \subseteq B(H_1 \hat{\otimes} H_2)$ *is a von Neumann algebra.*

(ii) *We have that*
$$(B(H_1) \hat{\otimes} M)' = \mathrm{id}_{H_1} \otimes M' \quad \textit{and} \quad (\mathrm{id}_{H_1} \otimes M)' = B(H_1) \mathbin{\overline{\otimes}} M'.$$
In particular, since $B(H_2)' = \mathbb{C}\mathrm{id}_{H_2}$,
$$(\mathrm{id}_{H_1} \otimes B(H_2))' = B(H_1) \otimes \mathrm{id}_{H_2} \quad \textit{and} \quad B(H_1) \mathbin{\overline{\otimes}} B(H_2) = B(H_1 \hat{\otimes} H_2).$$

Proof: Fix an orthonormal basis $(\xi_i)_{i\in I}$ of H_1. We define

- $v_{i,j} \in B(H_1)$ for $i, j \in I$ via $v_{i,j}\xi := \langle \xi, \xi_j\rangle_1 \xi_i$
- $V_i \in B(H_2, H_1 \hat{\otimes} B_2)$ for $i \in I$ via $V_{i,\eta} := \xi_i \otimes \eta$.

① *Claim:* $V_i V_j^* = v_{i,j} \hat{\otimes} \mathrm{id}_{H_2}$ *for all* $i, j \in I$.

Proof (of ①): First, we check that $V_j^*(\xi \otimes \eta) = \langle \xi, \xi_j\rangle \eta$. Indeed, for all $\tilde{\eta} \in H_2$,
$$\langle V_j\tilde{\eta}, \xi \otimes \eta\rangle = \langle \xi_j \otimes \tilde{\eta}, \xi \otimes \eta\rangle = \langle \xi_j, \xi\rangle_1 \langle \tilde{\eta}, \eta\rangle_2 = \langle \tilde{\eta}, \langle \xi, \xi_j\rangle_1 \eta\rangle_2 = \langle \tilde{\eta}, V_j^*(\xi \otimes \eta)\rangle.$$
Thus we get
$$V_i V_j^*(\xi \otimes \eta) = V_i(\langle \xi, \xi_j\rangle_1 \eta) = \langle \xi, \xi_j\rangle_1 \xi_1 \otimes \eta = v_{i,j}\xi \otimes \eta. \qquad \blacksquare$$

② *Claim:* $V_i^* V_j = \delta_{i,j} \mathrm{id}_{H_2}$.

Proof (of ②): Indeed we have
$$V_i^* V_j \eta = V_i^*(\xi_j \otimes \eta) = \langle \xi_j, \xi_i\rangle_1 \eta = \delta_{i,j}\eta. \qquad \blacksquare$$

③ *Claim: Let* $A := \{V_i \times V_j^* \mid i, j \in I, x \in M\}$, *then* $A' = \mathrm{id}_{H_1} \otimes M'$.

Proof (of ③): Note that $V_i \times V_j^* = v_{i,j} \hat{\otimes} x$, thus "$\supseteq$" is clear. Conversely, take $w \in A'$. Then, in particular
$$w V_i V_j^* = V_i V_j^* w$$
for all $i, j \in I$. Assertion 2 implies that $y := V_i^* w V_i = V_j^* w V_j$ for all $i, j \in I$ and $V_i^* w V_j = 0$ for all $i, j \in I$ where $i \neq j$; hence
$$w = \Big(\sum_{i\in I} V_i V_i^*\Big) w \Big(\sum_{j\in I} V_j V_j^*\Big) = \sum_{i\in I} V_j y V_i^* = \sum_{i\in I} v_{i,i} \hat{\otimes} y = \mathrm{id}_{H_1} \otimes y.$$
We want $y \in M'$: Take $x \in M$, then, for any $i \in I$,
$$yx \overset{②}{=} V_i^* w (V_i \times V_i^*) V_i = V_i^* (V_i \times V_i^*) w V_i = x V_i^* w V_i = xy. \qquad \blacksquare$$

④ *Claim:* $(B(H_1)\mathbin{\overline{\otimes}} M)' = \mathrm{id}_{H_1} \otimes M'$.

Proof (of ④): We clearly have $A \subseteq B(H_1)\mathbin{\overline{\otimes}} M$; thus

$$(B(H_1 \mathbin{\overline{\otimes}} M)' \subseteq A' \overset{③}{=} \mathrm{id}_{H_1} \otimes M' \subseteq (B(H_1)\mathbin{\overline{\otimes}} M)'$$

and hence $(B(H_1)\mathbin{\overline{\otimes}} M)' = \mathrm{id}_{H_1} \otimes M'$. ■

⑤ *Claim:* $(\mathrm{id}_{H_1} \otimes M)' = B(H_1)\mathbin{\overline{\otimes}} M'$.

Proof (of ⑤): Applying ④ to M' instead of M, we get that

$$(\mathrm{id}_{H_1} \otimes M)' = (B(H_1)\mathbin{\overline{\otimes}} M')'' = B(H_1)\mathbin{\overline{\otimes}} M'. \quad ■$$

Now, ④ and ⑤ prove (ii). For proving (i), note that by (ii) $\iota(M) = \mathrm{id}_{H_1} \otimes M = (B(H_1)\mathbin{\overline{\otimes}} M)'$ so that $\iota(M)$ is weakly closed (see (Lemma 6.7) (i)). Thus $\iota(M) = \mathrm{cl}_{\mathrm{WOT}}(M) = \iota(M)''$ and furthermore $\mathrm{cl}_{\mathrm{WOT}}(M) = (\mathbb{C}\mathrm{id}_{H_1})\mathbin{\overline{\otimes}} M$. ■

Example 10.3: If $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ are complex Hilbert spaces, then $\mathrm{id}_{H_1} \otimes B(H_2) \subseteq B(H_1 \hat{\otimes} H_2)$ is a factor of type I. Indeed:

$$Z(\mathrm{id}_{H_1} \otimes B(H_2)) = (\mathrm{id}_{H_1} \otimes B(H_1)) \cap (\mathrm{id}_{H_1} \otimes B(H_2))' = \mathbb{C}\mathrm{id}_{H_1 \hat{\otimes} H_2},$$

and $\mathrm{id}_{H_1} \hat{\otimes}\, e$ for any projection $e \in B(H_2)$ with $\dim eH = 1$ is minimal.

We prove now that in fact any factor of type I is of this particular form.

Theorem 10.4: *Let $M \subseteq B(H)$ be a factor of type I. Then there are Hilbert spaces H_1, H_2 and a unitary $U\colon H \to H_1 \hat{\otimes} H_2$ such that*

$$UMU^* = \mathrm{id}_{H_1} \otimes B(H_2) \quad \textit{and} \quad UM'U^* = B(H_1) \otimes \mathrm{id}_{H_2}.$$

Thus, we have a WOT*-continuous* **-isomorphism*

$$\Phi\colon B(H_2) \longrightarrow M, \qquad x \longmapsto U^*(\mathrm{id}_{H_1} \otimes x)U.$$

Proof: Consider a minimal projection $e \in M$. Then, by Exercise 2 of Sheet 8, we find a family $(e_i)_{i\in I}$ of mutually orthogonal projections in M such that

$$e_i \sim e \,\forall\, i \in I \quad \text{and} \quad 1 = r + \sum_{i\in I} e_i$$

for some projection $r \in M$ with $r \precsim e$ but $r \not\sim e$.

Claim: $r = 0$. Take a partial isometry $u \in M$ such that $u^*u = r$ and $uu^* \le e$. By minimality of e, either $uu^* = 0$ or $uu^* = e$; the latter would give $r \sim e$, which is excluded, thus $uu^* = 0$, which then gives $u = 0$ and hence $r = 0$.

Thus $1 = \sum_{i\in I} e_i$, i. e.

$$H = \bigoplus_{i\in I} e_i H.$$

Take now partial isometries $u_i \in M$ such that $u_i^* u_i = e_i$ and $u_i u_i^* = e$ for all $i \in I$. Then

$$U\colon H \longrightarrow eH \hat{\otimes} \ell^2(I) \quad (= H_1 \hat{\otimes} H_2), \qquad \xi \longmapsto \sum_{i\in I} (u_i \xi) \otimes \delta_i,$$

where $(\delta_i)_{i\in I}$ is an orthonormal basis of $\ell^2(I)$, is a surjective isometry, hence a unitary. Note that

$$U^*(\eta \otimes \delta) = \sum_{j\in I} \langle \delta, \delta_j \rangle u_j^* \eta.$$

Now, take any $x \in M$ and put $\tilde{x} := UxU^*$. Then

$$\tilde{x}(\eta \otimes \delta) = U \sum_{j\in I} \langle \delta, \delta_j \rangle x u_j^* \eta = \sum_{i,j\in I} \langle \delta, \delta_j \rangle \underbrace{(u_i x u_j^* \eta)}_{=: x_{i,j} \in M} \otimes \delta_i.$$

Now, since $ex_{i,j}e = u_i u_i^* (u_i x u_j^*) u_j u_j^* = u_i x u_j^* = x_{i,j}$ and $eMe = \mathbb{C}e$ by (Remark 9.12) (ii), it follows that $x_{i,j} = \lambda_{i,j} e$ for some $\lambda_{i,j} \in \mathbb{C}$. Thus, we get

$$\begin{aligned}
\tilde{x}(\eta \otimes \delta) &= \sum_{i,j\in I} \langle \delta, \delta_j \rangle \lambda_{i,j} e\eta \otimes \delta_i \\
&= \sum_{i,j\in I} \langle \delta, \delta_j \rangle \lambda_{i,j} \eta \otimes \delta_i \qquad\qquad (\text{as } e\eta \in eH) \\
&= \eta \otimes \Lambda\delta
\end{aligned}$$

where $\Lambda \in B(\ell^2(I))$ is defined by $\Lambda\delta := \sum_{i,j\in I} \langle \delta, \delta_i \rangle \lambda_{i,j} \delta_i$, (i. e., $\Lambda \mathrel{\hat{=}} (\lambda_{i,j})_{i,j\in I}$); finally

$$UxU^* = \tilde{x} = \mathrm{id}_{eH} \hat{\otimes} \Lambda \in B(eH \hat{\otimes} \ell^2(I)),$$

i. e., $UMU^* \subseteq \mathrm{id}_{eH} \otimes B(\ell^2(I))$.

We also have the other inclusion "$\supseteq$", since each $\Lambda \in B(\ell^2(I))$ induces an operator

$$x = U^*(\mathrm{id}_{eH} \otimes \Lambda)U = \sum_{i,j\in I} \langle \Lambda\delta_i, \delta_j \rangle u_j^* u_i \in M$$

(as $u_j^* u_i \in M$) such that $UxU^* = \mathrm{id}_{eH} \hat{\otimes} \Lambda$. Thus in total we get

$$UMU^* = \mathrm{id}_{eH} \otimes B(\ell^2(I))$$

and (Lemma 10.2) (ii) yields that

$$UM'U^* = (UMU^*)' = (\mathrm{id}_{eH} \otimes \ell^2(I))' = B(eH) \otimes \mathrm{id}_{\ell^2(I)}. \qquad \blacksquare$$

Definition 10.5: Let $M \subseteq B(H)$ be a factor of type I. We say that

(i) M is of *type* I_n, if $M \cong B(H_2)$ where $\dim H_2 = n$.

(ii) M is of *type* I_∞, if $M \cong B(H_2)$ where $\dim H_2 = \infty$.

11 Group von Neumann algebras and type II factors

We present a general construction that produces von Neumann algebras starting from discrete groups. This will show, that there are indeed “interesting” examples of von Neumann algebras. In particular, we will prove existence of factors of type II.

Throughout the following, let G be a discrete group. We associate with G the Hilbert space

$$\ell^2(G) := \Big\{\xi\colon G \longrightarrow \mathbb{C} : \sum_{x\in G} |\xi(x)|^2 < \infty\Big\}$$

with the inner product given by $\langle \xi, \eta\rangle := \sum_{g\in G} \xi(x)\overline{\eta(x)}$.

Definition 11.1: (i) The *left regular representation of* G is given by

$$\lambda\colon G \longrightarrow B(\ell^2(G)), \qquad g \longmapsto \lambda_g,$$

where $(\lambda_g\xi)(x) := \xi(g^{-1}x)$.

(ii) The *right regular representation of* G is given by

$$\rho\colon G \longrightarrow B(\ell^2(G)), \qquad g \longmapsto \rho_g$$

where $(\rho_g\xi)(x) := \xi(xg)$.

Remark 11.2: (i) For each $y \in G$, we define $\delta_y \in \ell^2(G)$ by

$$\delta_y(x) := \begin{cases} 1, & \text{if } x = y,\\ 0, & \text{else.}\end{cases}$$

Then $(\delta_y)_{y\in G}$ forms an orthonormal basis of $\ell^2(G)$. Thus, we may write for any $\xi \in \ell^2(G)$:

$$\xi = \sum_{y\in G} \xi(y)\delta_y$$

and λ_g and ρ_g are given by continuous and linear extension of $\lambda_g\delta_y = \delta_{gy}$ and $\rho_g\delta_y = \delta_{yg^{-1}}$. Since $\lambda_g\lambda_{g^{-1}} = 1 = \lambda_{g^{-1}}\lambda_g$ and $\lambda_g^* = \lambda_{g^{-1}}$, all λ_g are unitaries on $\ell^2(G)$; the same holds for ρ_g.

Definition 11.3: (i) For $\xi, \eta \in \ell^2(G)$, we define their *convolution* by

$$\xi*\eta\colon G \longrightarrow \mathbb{C}, \qquad (\xi*\eta)(x) := \sum_{g\in G} \xi(g)\eta(g^{-1}x) = \sum_{g\in G} \xi(xg^{-1})\eta(g). \quad (11.1)$$

In fact, $|(\xi * \eta)(x)| \leq \|\xi\|\|\eta\|$ for all $x \in G$.

(ii) Let $\xi \in \ell^2(G)$. We put $D_\xi := \{\eta \in \ell^2(G) \mid \xi * \eta \in \ell^2(G)\}$ and define the unbounded linear operator

$$L_\xi \colon D_\xi \longrightarrow \ell^2(G), \qquad \eta \longmapsto \xi * \eta.$$

Analogously, $D'_\xi := \{\eta \in \ell^2(G) \mid \eta * \xi \in \ell^2(G)\}$ and

$$R_\xi \colon D'_\xi \longrightarrow \ell^2(G), \qquad \eta \longmapsto \eta * \xi.$$

Lemma 11.4: *For $\xi \in \ell^2(G)$ both L_ξ and R_ξ have closed graph in $\ell^2(G) \oplus \ell^2(G)$. In particular: If $D_\xi = \ell^2(G)$, then $L_\xi \in B(\ell^2(G))$, and analogously, if $D'_\xi = \ell^2(G)$, then $R_\xi \in B(\ell^2(G))$.*

Proof: Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^2(G)$ such that $\xi_n \to \eta \in \ell^2(G)$ and that $L_\xi \eta_n \to \zeta \in \ell^2(G)$. Then, for each $x \in G$, we have

$$\begin{aligned}|\zeta(x) - (\xi * \eta)(x)| &= \lim_{n\to\infty} |(\xi * \eta_n)(x) - (\xi * \eta)(x)| \\ &= \lim_{n\to\infty} |\xi * (\eta_n - \eta)(x)| \leq \lim_{n\to\infty} \|\xi\| \|\eta_n - \eta\| = 0,\end{aligned}$$

thus we get $\xi * \eta = \zeta \in \ell^2(G)$, i. e., $\eta \in D_\xi$ and $L_\xi \eta = \zeta$. By the closed graph theorem (Theorem 4.16 from the Functional Analysis I lecture notes), we see that $L_\xi \in B(\ell^2(G))$ if $D_\xi = \ell^2(G)$. ■

Definition 11.5: A vector $\xi \in \ell^2(G)$ is called

- *left-convolver* if $\xi * \ell^2(G) \subseteq \ell^2(G)$ (i. e., $D_\xi = \ell^2(G)$),
- *right-convolver* if $\ell^2(G) * \xi \subseteq \ell^2(G)$ (i. e., $D'_\xi = \ell^2(G)$).

We define

$$\begin{aligned}L(G) &:= \{L_\xi \mid \xi \in \ell^2(G) \text{ is left-convolver}\} \subseteq B(\ell^2(G)), \\ R(G) &:= \{R_\xi \mid \xi \in \ell^2(G) \text{ is right-convolver}\} \subseteq B(\ell^2(G)).\end{aligned}$$

Remark 11.6: (i) For each $g \in G$, $\delta_g \in \ell^2(G)$ is both a left— and right-convolver. We have (see Eq. (11.1)) $\delta_g * \xi = \lambda_g \xi$, i. e., $L_{\delta_g} = \lambda_g$ and analogously $\xi * \delta_g = \rho_{g^{-1}} \xi$, i. e., $R_{\delta_{g^{-1}}} = \rho_g$; thus, $\lambda(G) \subseteq L(G)$ and $\rho(G) \subseteq R(G)$.

(ii) For $\xi \in \ell^2(G)$, we define $\overline{\xi} \in \ell^2(G)$ by $\overline{\xi}(x) := \overline{\xi(x^{-1})}$. If $\xi \in \ell^2(G)$ is a left-convolver, then so is $\overline{\xi}$ and we have that $L_{\overline{\xi}} = L_\xi^*$. Similarly, $R_{\overline{\xi}} = R_\xi^*$ for right-convolvers ξ.

(iii) Since convolution is associative, we have $L_{\xi * \eta} = L_\xi L_\eta$ and $R_{\xi * \eta} = R_\eta R_\xi$.

Remark 11.6 shows that $L(G)$ and $R(G)$ are unital *-subalgebras of $B(\ell^2(G))$ that contain $\lambda(G)$ and $\rho(G)$ respectively. We next show, that actually both are von Neumann algebras and commutants of each other.

Theorem 11.7: *Let G be a discrete group. Then both $L(G)$ and $R(G)$ are von Neumann algebras acting on $\ell^2(G)$ and it holds true that*

$$L(G) = R(G)' = \rho(G)' \quad \text{and} \quad R(G) = L(G)' = \lambda(G)'$$

and therefore $\rho(G)'' = R(G)$ and $\lambda(G)'' = L(G)$.

We call $L(G)$ and $R(G)$ the left— *and* right group von Neumann algebra of G *repectively.*

Proof: ① *Claim: $\rho(G)' \subseteq L(G)$.*

Proof (of ①): Take $T \in \rho(G)'$ and put $\xi := T\delta_e \in \ell^2(G)$. Then, for all $g \in G$,

$$\begin{aligned} \xi * \delta_g &= \rho_{g^{-1}}\xi && \text{(by Eq. (11.1))} \\ &= \rho_{g^{-1}}T\delta_e = T\rho_{g^{-1}}T\delta_e = T\rho_{g^{-1}}\delta_e = T\delta_g \end{aligned}$$

and hence, by linearity, we get

$$\xi * \eta = T\eta$$

for all $\xi \in \langle\{\delta_g \mid g \in G\}\rangle \subseteq \ell^2(G)$. If $\eta \in \ell^2(G)$ is given, we find a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\langle\{\delta_g \mid g \in G\}\rangle \subseteq D_\xi$ such that $\eta_n \to \eta$ and $L_\xi\eta_n = T\eta_n \to T\eta$ (due to the continuity of T). Thus, by Lemma 11.4, $\eta \in D_\xi$ and $L_\xi\eta = T\eta$. Therefore $D_\xi = \ell^2(G)$, i. e., ξ is a left-convolver and $T = L_\xi \in L(G)$. ■

② *Claim: $L(G) = R(G)' = \rho(G)'$.*

Proof (of ②): We clearly have $L(G) \subseteq R(G)' \subseteq \rho(G)'$ and by ① we also have $\rho(G)' \subseteq L(G)$. In particular, $L(G)$ is weakly closed (by (Lemma 6.7) (i)) and hence a von Neumann algebra. ■

③ Similarly, $R(G)$ is a von Neumann algebra and $R(G) = L(G)' = \lambda(G)'$.

④ Using ② and ③ and von Neumanns bicommutant theorem (Theorem 6.8), we infer that

$$\rho(G)'' \overset{②}{=} R(G)'' \underset{6.8}{\overset{③}{=}} R(G) \quad \text{and} \quad \lambda(G)'' \overset{③}{=} L(G)'' \underset{6.8}{\overset{②}{=}} L(G).$$

■

Theorem 11.8: *Let G be a discrete group. Then*

$$\tau \colon L(G) \longrightarrow \mathbb{C}, \qquad x \longmapsto \langle x\delta_e, \delta_e\rangle$$

is a positive linear functional with $\tau(\mathrm{id}) = 1$ (i. e., a state), which is WOT-*continuous and moreover*

- faithful, *i. e., "$\tau(x^*x) = 0 \Rightarrow x = 0$",*
- tracial, *i. e., $\tau(xy) = \tau(yx)$ for all $x, y \in L(G)$.*

Proof: We only have to verify, that τ is faithful and tracial; the other assertions are obvious.

① *Claim: τ is faithful.*

Proof (of ①): Take $x = L_\xi \in L(G)$ for some left-convolver $\xi \in \ell^2(G)$. Then:

$$\|\xi\|^2 = \|L_\xi \delta_e\|^2 = \langle L_\xi \delta_e, L_\xi \delta_e \rangle = \langle L_\xi^* L_\xi \delta_e, \delta_e \rangle = \tau(x^* x).$$

Thus, if $\tau(x^*x) = 0$ implies $\xi = 0$ and hence $x = L_\xi = 0$. ■

② *Claim: τ is tracial.*

Proof (of ②): By the WOT-continuity of τ, it suffices to verify that $\tau(xy) = \tau(yx)$ for all x, y in the strongly dense subalgebra $\langle \{\lambda_g \mid g \in G\} \rangle$ (due to Theorem 11.7). By the linearity, we only have to check that $\tau(\lambda_g \lambda_h) = \tau(\lambda_h \lambda_g)$ for all $g, h \in G$. This can be seen as follows:

$$\begin{aligned} \tau(\lambda_g \lambda_h) = \langle \lambda_g \lambda_h \delta_e, \delta_e \rangle = \langle \delta_{gh} \delta_e, \delta_e \rangle &= \begin{cases} 1, & gh = e \\ 0, & \text{else.} \end{cases} \\ &= \begin{cases} 1, & hg = e, \\ 0, & \text{else.} \end{cases} = \langle \lambda_h \lambda_g \delta_e, \delta_l \rangle = \tau(\lambda_h \lambda_g) \end{aligned}$$

which concludes the proof of ②. ■

As τ is indeed tracial and faithful, we have proven Theorem 11.8. ■

Which (left) group von Neumann algebras are factors?

Definition 11.9: A group G is called *infinite conjugacy class group* (abbreviated *i.c.c. group*), if each non-trivial (i. e., $h \neq e$) conjugacy class $\{ghg^{-1} \mid g \in G\}$ is infinite.

Theorem 11.10: *Let G be a discrete group. Then $L(G)$ is a factor if and only if G is an infinite conjugacy class group.*

Proof: "$\Rightarrow$": Suppose G is *not* an infinite conjugacy class group, i. e., we find a non-trivial element $e \neq h \in G$ such that $h^G := \{ghg^{-1} \mid g \in G\}$ is finite. We define $x := \sum_{k \in h^G} \lambda_k \in L(G)$. Since $gh^G h^{-1} = h^G$ for all $g \in G$, we have

$$\lambda_g x \lambda_{g^{-1}} = \sum_{k \in h^G} \lambda_{gkg^{-1}} = \sum_{k \in h^G} \lambda_k = x$$

and thus $\lambda_g x = x \lambda_g$. By Theorem 11.7, we have that $x \in \lambda(G)' = L(G)'$; hence $x \in L(G) \cap L(G)' = Z(L(G))$. Now, observe that $x \notin \mathbb{C}1$ as $(\lambda_g)_{g \in G}$ are linearly independent in $L(G)$ and $\lambda_e = 1$ — Indeed, if $g_1, \dots, g_n \in G$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ with $\sum_{i=1}^n \alpha_i \lambda_i = 0$ and $g_1, \dots, g_n$ mutually different are given, then

$$0 = \Big(\sum_{i=1}^n \alpha_i \lambda_{g_i}\Big)\delta_e = \sum_{i=1}^n \alpha_i \delta_{g_i},$$

and since $(\delta_g)_{g\in G}$ is an orthonormal basis of $\ell^2(G)$, we infer $\alpha_1 = \dots = \alpha_n = 0$. Thus $Z(L(G)) \supsetneq \mathbb{C}1$, i.e., $L(G)$ is not a factor.

"$\Leftarrow$": Suppose that G is an infinite conjugacy class group. Take any $x \in Z(L(G))$; since $x \in L(G)$, we find a left-convolver $\xi \in \ell^2(G)$ such that $x = L_\xi$ and since also $x \in L(G)'$, it satisfies for $h \in G$:

$$L_\xi = x = \lambda_{h^{-1}} x \lambda_h = \lambda_h^* x \lambda_h = L_{\delta_{h^{-1}} * \xi},$$

and if applied to δ_e, $\xi = \delta_{h^{-1}} * \xi * \delta_h$. Thus, for any $g \in G$:

$$\xi(g) = ((\delta_{h^{-1}} * \xi) * \delta_h)(g) \overset{\text{Eq. (11.1)}}{=} (\delta_{h^{-1}} * \xi)(gh^{-1}) \overset{\text{Eq. (11.1)}}{=} \xi(hgh^{-1}).$$

Thus, $\xi\colon G \to \mathbb{C}$ is constant on $\{hgh^{-1} \mid h \in G\}$ for each $g \in G$. Since G is assumed to be an infinite conjugacy class group and $\xi \in \ell^2(G)$, $\xi \equiv 0$ is enforced on $\bigcup_{g \in G\setminus\{e\}} \{hgh^{-1} \mid h \in G\} = G\setminus\{e\}$. Hence: $\xi = \xi(e)\delta_e$ and thus $x = \xi(e)1 \in \mathbb{C}1$.■

What can be said about the type of such factors?

Remark 11.11: (i) A von Neumann algebra $M \subseteq B(H)$ is said to be *finite* if $1 \in M$ is a finite projection, or equivalently, if every isometry in M is a unitary (i.e., $v^*v = 1 \Rightarrow vv^* = 1$). According to Exercise 2 (a), Sheet 7, each projection in a finite von Neumann algebra is finite.

(ii) Let $M \subseteq B(H)$ be a von Neumann algebra, that has a faithful tracial state $\tau\colon M \to \mathbb{C}$, which is moreover *normal*, i.e., WOT-continuous on

$$M_1 := \{x \in M \mid \|x\| \le 1\}.$$

Then M is finite (In fact, the converse is also true, this requires work).

(iii) Let $M \subseteq B(H)$ be a factor of type I. Then M is finite if and only if M is of type I_n for some $n \in \mathbb{N}$, which follows from Theorem 10.4 and (Remark 9.12) (i). In particular, a factor of type I is finite if and only if it is finite dimensional.

Definition 11.12: Let $M \subseteq B(H)$ be a factor of type II. We say that

(i) *M is of type II_1*, if 1 is a finite projection.

(ii) *M is of type II_∞*, if 1 is not a finite projection (but there are finite projections).

Remark 11.13: Let $M \subseteq B(H)$ be a factor. If M has a faithful normal tracial state $\tau\colon M \to \mathbb{C}$ and is not finite dimensional, then M is of type II_1. Indeed: 1 is finite due to Remark 11.11 (ii) but not of type I due to Remark 11.11 (iii), hence of type II_1.

Corollary 11.14: *Let G be a non-trivial discrete infinite conjugacy class group. Then $L(G)$ is a factor of type II_1.*

Proof: By Theorem 11.10, $L(G)$ is a factor. Due to Theorem 11.8, it admits a faithful normal tracial state $\tau\colon L(G) \to \mathbb{C}$. Since G is non-trivial and an infinite conjugacy class group, it contains infinitely many elements; thus $L(G)$ is not finite dimensional. The assertion now follows from Remark 11.13. ■

Example 11.15: (i) Consider $G = S_\infty := \bigcup_{n\in\mathbb{N}} S_n$, i. e., the group of all permutations of $\mathbb{N}$ that move only finitely many points. S_∞ is an infinity conjugacy class group. Indeed, if $\sigma \in S_\infty$, $\sigma \neq \mathrm{id}$ is given, then we find $i \neq j$ such that $\sigma(i) = j$. Consider the transposition $\pi_r := (i, r)$ for $r > \max\{i, j\}$, then we have

$$(\pi_r \sigma \pi_r^{-1})(r) = (\pi_r \sigma \pi_r)(r) = j,$$

i. e., all $\pi_r \sigma \pi_r^{-1}$ are different. Thus $L(S_\infty)$ is a type II_1 factor. It is the so-called *hyperfinite II_1-factor $\mathcal{R}$*.

(ii) Consider $G = \mathbf{F}_n$, the free group with n generators; if $n \geq 2$, then $\mathbf{F}_n$ is an infinite conjugacy class group. Thus, $L(\mathbf{F}_n)$ is a type II_1 factor; a so-called *free group factor*.

Murray and von Neumann have shown, that $L(\mathbf{F}_n) \ncong \mathcal{R}$; for that purpose they have introduced the so-called *property* Γ. It is still an open problem, whether $L(\mathbf{F}_n) \cong L(\mathbf{F}_m)$ for $n, m \geq 2$, $n \neq m$. This problem motivated Voiculescu (arround 1985) to develop "free probability theory"; this has led to discover, in particular, amazing connections between such operator algebraic questions and random matrix theory. By using that bridge, Dykema and Radulescu (1994) have shown independently that

- either $L(\mathbf{F}_n) \cong L(\mathbf{F}_m)$ for all $n, m \geq 2$,
- or $L(\mathbf{F}_n) \ncong L(\mathbf{F}_m)$ for all $n, m \geq 2$, $n \neq m$.

What other infinite conjugacy class groups G satisfy $L(G) \cong \mathcal{R}$?

Remark 11.16: (i) A separable von Neumann algebra $M \subseteq B(H)$ is called hyperfinite, if there is an increasing sequence $(A_n)_{n\in\mathbb{N}\setminus\{0\}}$ of finite dimensional *-subalgebras of M such that

$$M = \mathrm{cl}_{\mathrm{WOT}}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big).$$

For $M = L(S_\infty)$, one can choose $A_n := L(S_n)$.

(ii) It was proven by Murray and von Neumann that up to isomorphism, there is a unique hyperfinite II_1-factor; it is denoted by $\mathcal{R}$.

(iii) One can show that $\mathcal{R}$ embeds into any factor of type II_1. On the other hand, the *Connes-embedding conjecture problem* asks whether every type II_1-factor on a separable Hilbert space can be embedded into the ultrapower $\mathcal{R}^\omega$ of $\mathcal{R}$ by a free ultra-filter ω; roughly speaking, this asks for matricial approximations.

Definition 11.17: A countable discrete group is called *amenable*, if there is a state $m\colon \ell^\infty(G) \to \mathbb{C}$ such that for all $f \in \ell^\infty(G)$ and $g \in G$ it holds

$$m(f) = m(\lambda g f)$$

where $(\lambda g f)(x) := f(g^{-1}x)$ for all $x \in G$; m is called a *left-invariant mean on* G.

Remark 11.18: (i) If G is a finite group, then

$$m(f) := \frac{1}{|G|} \sum_{g \in G} f(g)$$

defines a left-invariant mean on G.

(ii) If G can be written as $G = \bigcup_{n \in \mathbb{N}} G_n$ for an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of subgroups of G, then it holds: If G_n is amenable for all $n \in \mathbb{N}$, then so is G. This shows that $S_\infty = \bigcup_{n \in \mathbb{N}} S_n$ is amenable.

As a consequence of "Connes' tour de force" (1976) about *injective* type II_1 factors, we have:

Theorem 11.19: *Let G be a countable non-trivial infinite conjugacy class group. Then $L(G) \cong \mathcal{R}$ if and only if G is amenable.*

12 The trace construction on finite factors

In this Chapter, we briefly discuss how on finite factors, i. e., factors that finite in the sense of (Remark 11.11) (i), a unique faithful normal tracial state can be constructed; see (Remark 11.11) (ii). The main results read as follows:

Theorem 12.1 (Existence of the trace): *Let $M \subseteq B(H)$ be a factor. Then the following are equivalent:*

(i) *M is finite.*

(ii) *M has a (norm-continuous) tracial state $\tau\colon M \to \mathbb{C}$.*

(iii) *M has a normal tracial state $\tau\colon M \to \mathbb{C}$.*

Theorem 12.2 (Uniqueness of the trace): *Let $M \subseteq B(H)$ be a finite factor. Then there is a unique (norm-continuous) tracial state $\tau\colon M \to \mathbb{C}$. Moreover, τ is automatically normal and faithful.*

Every finite factor M is either of type I_n for some $n \in \mathbb{N}$ or of type II_1; if M is of type I_n, then $M \cong B(\mathbb{C}^n) = M_n(\mathbb{C})$. Then the trace τ on M comes from

$$\mathrm{tr}_n\colon M_n(\mathbb{C}) \longrightarrow \mathbb{C}, \qquad (a_{i,j})_{1\le i,j\le n} \longmapsto \frac{1}{n}\sum_{i=1}^{n} a_{i,i}.$$

Thus, it suffices to treat the type II_1 case.

Theorem 12.3 (Dimension function): *Let $M \subseteq B(H)$ be a factor of type II_1. Then there exists a function*

$$\Delta\colon \mathcal{P}(M) \longrightarrow [0,1],$$

the so-called dimension function of M*, with the following properties:*

(i) *$\Delta(1) = 1$ and for all $p \in \mathcal{P}(M)$ it holds $\Delta(p) = 0$ if and only if $p = 0$.*

(ii) *For all $p, q \in \mathcal{P}(M)$ it holds $p \precsim q$ if and only if $\Delta(p) \le \Delta(q)$. In particular it holds $p \sim q$ if and only if $\Delta(p) = \Delta(q)$.*

(iii) *For all $p, q \in \mathcal{P}(M)$, $pq = 0$, it holds $\Delta(p+q) = \Delta(p) + \Delta(q)$.*

(iv) *Δ is* completely additive*, i. e., for each family $(p_i)_{i\in I}$ of mutually orthogonal projections in M, it holds $\Delta(\sum_{i\in I} p_i) = \sum_{i\in I} \Delta(p_i)$.*

Proof (Sketch): ① "*Halving lemma*": For each $p \in \mathcal{P}(M)$, there are projections $p_0, p_1 \in \mathcal{P}(M)$ such that $p_0 \sim p_1$ and $p_0 + p_1 = p$; in other words, "p_0 and p_1 halve p".

Proof (of ①): (a) Since p is not minimal, we find $q \in \mathcal{P}(M)$ such that $q \leq p$, $q \neq 0$, and $q \neq p$. Since M is a factor, Claim ① in the proof of (Lemma 9.9) shows, that $qM(p-q) \neq \{0\}$. Take $x \in M$ with $y := qx(p-q) \neq 0$ and consider its polar decomposition $y = u|y|$, then $u \neq 0$ and $u^*u + uu^* \leq p$; clearly $u^*u \perp uu^*$. Indeed, note that u^*u is the projection onto $\ker(y)^\perp \subseteq (p-q)H \subseteq (1-q)H$ and uu^* is the projection onto $\mathrm{cl}(\mathrm{im}(y)) \subseteq qH$.

(b) Consider families $(p_i, q_i)_{i \in I}$ such that

- $(p_i)_{i\in I}$ are mutually orthogonal projections in M,
- $(q_i)_{i\in I}$ are mutually orthogonal projections in M,
- For all $i \in I$ it holds $p_i \sim q_i$,
- $\sum_{i\in I} p_i + \sum_{i \in I} q_i \leq p$.

By Zorn's Lemma, there is a maximal family $(p_i, q_i)_{i\in I}$ of this kind; now put $p_0 := \sum_{i\in I} p_i$ and $q_0 := \sum_{i\in I} q_i$. Using (a), it follows due to maximality of $(p_i, q_i)_{i\in I}$ that $p_0 + q_0 = p$. ■

Note: The halving lemma is true in any *diffuse* factor; a von Neumann algebra is said to be *diffuse*, if it contains no minimal projections.

② "*Fundamental projections*": There is a sequence $(p_n)_{n\in\mathbb{N}\setminus\{0\}}$ of mutually orthogonal projections on M such that for all $n \in \mathbb{N}$ it holds

$$p_n \sim 1 - \sum_{i=1}^{n} p_i$$

One constructs the sequence $(p_n)_{n\in\mathbb{N}}$ by iterating ①. We have the following:

(a) If for $p \in \mathcal{P}(M)$ it holds $p \precsim p_n$ for all $n \in \mathbb{N}$, then $p = 0$.

(b) $\sum_{n\in\mathbb{N}} p_n = 1$ and therefore $p_n \sim \sum_{i \geq n+1} p_i$.
Indeed: $p := 1 - \sum_{i\in\mathbb{N}} p_i \leq 1 - \sum_{i=1}^{n} p_i \sim p_n$ for all $n \in \mathbb{N}$, by (a) now $p = 0$ follows.

(c) For all $0 \neq p \in \mathcal{P}(M)$ there is $n \in \mathbb{N}$ such that $p_n \precsim p$.

Define

$$\mathcal{FP}(M) := \{p \in \mathcal{P}(M) \mid \exists\, n \in \mathbb{N} : p \sim p_n\},$$

the set of *fundamental projections in M*

Fundamental projections (or, more precisely, equivalence classes thereof) play for projections the role of dyadic rationals for numbers in $[0,1]$. In fact, we have: For each $0 \neq p \in M$, there exists a unique increasing sequence $n_1 < n_2 < n_3 < \dots$ in $\mathbb{N}$ and a sequence $(p'_k)_{k\in\mathbb{N}}$ of mutually orthogonal projections in M such that

(a) $p'_k \sim p_k$ for all $k \in \mathbb{N}$,

(b) $p = \sum_{k\in\mathbb{N}} p'_k$.

We define

$$\Delta(p) := \sum_{k\in\mathbb{N}} 2^{-n_k},$$

i. e., $\Delta(p_n) = 2^{-n}$. One can show — but this requires work — that this yields the desired dimension function. ■

Proof (of Theorem 12.1): ① "Radon-Nikodym-trick": Let $\varphi, \psi\colon \mathcal{P}(M) \to [0,\infty)$ be completely additive maps with $\varphi, \psi \not\equiv 0$. Suppose that φ is faithful, i. e., $\varphi(e) \neq 0$ whenever $e \neq 0$. Then

$$\forall\, \varepsilon > 0\, \exists\, p \in \mathcal{FP}(M), \theta > 0\, \forall\, p \in \mathcal{P}(M), q \leq p : \theta\varphi(q) \leq \psi(q) \leq \theta(1+\varepsilon)\varphi(q).$$

② Let $\psi\colon M \to \mathbb{C}$ be a positive linear functional and $\varepsilon > 0$ such that

$$\forall\, p \in \mathcal{P}(M) : \Delta(q) \leq \psi(q) \leq (1+\varepsilon)\Delta(q).$$

Then for all positive $x \in M$ and unitaries $u \in M$,

$$\psi(uxu^*) \leq (1+\varepsilon)\psi(x),$$

i. e., ψ is an ε-*trace*; equivalently, ψ satisfies $\psi(xx^*) \leq (1+\varepsilon)\psi(x^*x)$ for all $x \in M$.

③ Using ① and ②, one can show that for every $\varepsilon > 0$ a normal ε-trace ψ_ε exists with

$$\forall\, q \in \mathcal{P}(M) : (1+\varepsilon)^{-1}\Delta(q) \leq \psi_\varepsilon(q) \leq (1+\varepsilon)^2\Delta(q).$$

④ Choose a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$; then for all $p \in \mathcal{P}(M)$ it holds

$$\lim_{n\to\infty} \psi_{\varepsilon_n}(q) = \Delta(q).$$

and in fact, $(\psi_{\varepsilon_n})_{n\in\mathbb{N}}$ converges in norm to a linear functional $\psi\colon M \to \mathbb{C}$ which turns out to be a faithful normal tracial state. This shows: "(i) $\Rightarrow$ (iii)" holds also for M being of type II_1.

"(iii) $\Rightarrow$ (ii)" is trivial and "(ii) $\Rightarrow$ (i)" is an exercise. ■

Proof (of Theorem 12.2): ① "*Dixmier's averaging theorem*": Let $M \subseteq B(H)$ be a factor and let $x \in M$ be given. Then

(a) For all $\varepsilon > 0$ there are unitaries $u_1, \ldots, u_n \in M$ and $\alpha \in \mathbb{C}$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i x u_i^* - \alpha 1 \right\| < \varepsilon.$$

(b) We put $K_x := \mathrm{cl}_{\|\cdot\|}(\mathrm{conv}\{uxu^* \mid u \in M \text{ unitary}\})$; then $K_x \cap \mathbb{C}1 \neq \varnothing$.

Note: (b) follows from (a) as $\varepsilon \downarrow 0$.

② If $\tau\colon M \to \mathbb{C}$ is a norm-continuous tracial state on the factor M, then $K_x \cap \mathbb{C}1 = \{\tau(x)1\}$. Thus, there is at most one norm-continuous tracial state on M; hence the one found in Theorem 12.1 the unique one, which is even normal and faithful. ■

Remark 12.4: Consider $M = M_n(\mathbb{C})$. Then

$$\operatorname{tr}_n(x)1 = \int_{U_n(\mathbb{C})} uxu^* \, du,$$

where $U_n(\mathbb{C}) = \{u \in M_n(\mathbb{C}) \mid u \text{ unitary}\}$ and "du" stands for the Haar probability measure on $U_n(\mathbb{C})$. This fact is generalised by ① (b) in the proof of Theorem 12.2.

13 The standard representation of tracial von Neumann algebras

Motivation 13.1: So far, we have studied von Neumann algebras (mostly) on their "own" Hilbert space that accompanies them by definition. However, especially for *tracial von Neumann algebras*, i. e., von Neumann algebras that are equipped with a faithful normal tracial state (which is unique in the case of a factor due to Theorem 12.2), there are other, sometimes better behaved, representations on other Hilbert spaces.

Definition 13.2: Let $M \subseteq B(H)$ be a von Neumann algebra and let K be another complex Hilbert space.

(i) A unital *-homomorphism $\pi\colon M \to B(K)$ is called a *representation of M on K* (see Definition 5.12).

(ii) A representation $\pi\colon M \to B(K)$ is said to be *normal*, if it's restriction to $M_1 = \{x \in M \mid \|x\| \leq 1\}$ is WOT-continuous.

(iii) A representation $\pi\colon M \to B(K)$ is said to be *faithful*, if it is injective.

Theorem 13.3: *Let $\pi\colon M \to B(K)$ be a normal representation of M on K. Then $\pi(M) \subseteq B(K)$ is a von Neumann algebra.*

We do not give a proof but we point out that special cases appeared before in

- the proof of Theorem 8.15,
- in (Lemma 10.2) (i).

Theorem 13.4: *Let $M \subseteq B(H)$ be a von Neumann algebra and let $\varphi\colon M \to \mathbb{C}$ be a state. Consider the cyclic representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ obtained by the GNS construction with respect to φ, see Theorem 5.15. Then φ is normal (in the sense of (Remark 11.11) (ii)) if and only if $\pi_\varphi\colon M \to B(H_\varphi)$ is normal.*

Proof: Since $\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle$ for all $x \in M$, π_φ being normal enforces φ to be normal.

Conversely, if φ is normal, then

$$x \longmapsto \langle \pi_\varphi(x)\hat{a}, \hat{b}\rangle_\varphi = \varphi(b^*xa)$$

is WOT-continuous on M_1 for all $a, b \in M$, where $\hat{a}$ denotes the class of a in $M/N_\varphi \subseteq H_\varphi$; note that $\xi_\varphi = \hat{1}$, so that $\hat{a} = \pi_\varphi(a)\xi_p$. Now, since ξ_φ is cyclic, i. e., $\pi_\varphi(M)\xi_\varphi$ is dense in H_φ, we infer that $x \mapsto \langle \pi_\varphi(x), \eta\rangle_\varphi$ is WOT-continuous on M_1 for all $\xi, \eta \in H_\varphi$. Thus, π_φ is normal. ■

Definition 13.5: Let (M, τ) be a tracial von Neumann algebra. We call the representation $\pi_\tau \colon M \to B(H_\tau)$ the *standard representation of* (M, τ) and we write

$$L^2(M, \tau) := H_\tau \ (= \mathrm{cl}_{\langle \cdot, \cdot \rangle_\tau}(M) \text{ as } N_\tau = \{0\}).$$

The standard representation is normal (Theorem 13.4) and moreover faithful, since $\tau(x) = \langle \pi_\tau(x)\xi_\tau, \xi_\tau \rangle_\tau$ for all $x \in M$, thus we may identify M with its image $\pi_\tau(M) \subseteq B(L^2(M, \tau))$, in which case M is said to be in *standard form*. Note that $\xi_\tau = \hat{1}$ is both cyclic and separating for M.

Remark 13.6: (i) A von Neumann algebra is said to be *separable*, if it has a faithful normal representation on a separable Hilbert space. This generalises (Definition 8.6), without changing the conclusions.

(ii) Let (M, τ) be a tracial von Neumann algebra. Then (M_1, d) with

$$d(x, y) := \|\hat{x} - \hat{y}\|_\tau$$

is a complete metric space and the induced topology conincides with the strong operator topology from $B(L^2(M, \tau))$. For a tracial von Neumann algebra (M, τ), the following are equivalent:

- M is separable,
- M_1 is separable contains a SOT-dense sequence,
- $L^2(M, \tau)$ is a separable Hilbert space.

Example 13.7: Consider the tracial von Neumann algebra $(M_n(\mathbb{C}), \mathrm{tr}_n)$ acting on $\mathbb{C}^n$. It is in standard form on $L^2(M_n(\mathbb{C}), \mathrm{tr}_n) = \mathbb{C}^{n^2} \cong \mathbb{C}^n \otimes \mathbb{C}^n$ with

$$\begin{aligned} \pi_{\mathrm{tr}_n}(M_n(\mathbb{C})) &= M_n(\mathbb{C}) \otimes 1 \subseteq B(\mathbb{C}^n \otimes \mathbb{C}^n) \\ \text{and } \pi_{\mathrm{tr}_n}(M_n(\mathbb{C}))' &= 1 \otimes M_n(\mathbb{C}) \subseteq B(\mathbb{C}^n \otimes \mathbb{C}^n), \end{aligned}$$

where $\pi_{\mathrm{tr}_n}(x) = x \otimes 1$.

Definition 13.8: Consider a tracial von Neumann algebra (M, τ). The antilinear unitary involution $J \colon L^2(M, \tau) \to L^2(M, \tau)$ which is the extension to $L^2(M, \tau)$ of the antiunitary isometry

$$J\hat{x} := \widehat{x^*}$$

for all $x \in M$, the *canonical conjugation operator on* $L^2(M, \tau)$. It satisfies $J^2 = 1$ and $\langle J\xi, \eta \rangle_\tau = \langle J\eta, \xi \rangle_\tau$ for all $\xi, \eta \in L^2(M, \tau)$.

Theorem 13.9: *Let (M, τ) be a tracial von Neumann algebra in standard form on $L^2(M, \tau)$. Then $JMJ = M'$ on $B(L^2(M, \tau))$ and $\tau_{M'}(x) := \langle x\xi_\tau, \xi_\tau \rangle_\tau$ defines a faithful normal tracial state on M'.*

Remark 13.10: For $x \in M$ we have $L_x := \pi_\tau(x) \in B(L^2(M, \tau))$ with $L_x\hat{y} = \widehat{xy}$ for all $y \in M$. Similarly, we have $R_x \in B(L^2(M, \tau))$ with $R_x\hat{y} = \widehat{yx}$ for all $y \in M$. Then $JL_xJ = R_{x^*}$. Hence $M' = JMJ = \{R_x \mid x \in M\}$.

Definition 13.11: Let M be a von Neumann algebra.

(i) A *(left) M-module* is a Hilbert space H that comes with a normal representation $\pi\colon M \to B(H)$. We write (H, π) or ${}_M H$.

(ii) Let (H, π_H) and (K, π_K) be two M-modules. A *(left) modular operator from H to K* is an operator $T \in B(H, K)$ such that for all $x \in M$ it holds

$$T\pi_H(x) = \pi_K(x)T.$$

In other words: The diagram

$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, "\pi_H(x)"'] & K \arrow[d, "\pi_K(x)"] \\ H \arrow[r, "T"'] & K \end{tikzcd}$$

commutes. The space of all modular operators from H to K will be denoted by ${}_M B(H, K)$.

(iii) We say that ${}_M H$ and ${}_M K$ are *isomorphic* (or *equivalent*), ${}_M H \cong {}_M K$, if there exists a unitary operator in ${}_M B(H, K)$.

Example 13.12: Let (M, τ) be a tracial von Neumann algebra.

(i) The standard representation of (M, τ) gives a "canonical" M-module $L^2(M, \tau)$; see (Definition 13.5).

(ii) We can produce "larger" M-modules by amplifications: $H = L^2(M, \tau) \mathbin{\hat{\otimes}} K$ for some (arbitrary) Hilbert space K; indeed

$$\pi\colon M \longrightarrow B(H), \qquad \pi(x)(\xi \otimes \eta) := (x\xi) \otimes \eta$$

defines a normal representation. For example

$$L^2(M, \tau) \mathbin{\hat{\otimes}} \mathbb{C}^n \cong \bigoplus_{i=1}^{n} L^2(M, \tau)$$

and $\pi(x)(\xi_1, \dots, \xi_n) := (x\xi_1, \dots, x\xi_n)$.

(iii) For an M-module (H, π_H), we can construct a "smaller" M-module by "cutting down" with any projection $p \in {}_M B(H)$, i. e., we take the Hilbert space pH with the normal representation

$$\pi_{pH}\colon M \longrightarrow B(pH), \qquad x \longmapsto \pi_H(x)|_{pH} \in B(pH).$$

Note that ${}_M BH = \pi_H(M)'$.

By combining these constructions, we obtain in fact all separable M-modules of a separable type II_1 factor M.

Theorem 13.13: *Let M be a separable factor of type II_1 in standard form on $L^2(M,\tau)$ for its unique faithful normal tracial state $\tau\colon M \to \mathbb{C}$. For every separable M-module ${}_MH$, there exists an isometry $v \in {}_MB(H, L^2(M,\tau) \otimes \ell^2(\mathbb{N}))$. In particular, $p := vv^* \in {}_MB(L^2(M,\tau) \hat{\otimes} \ell^2(\mathbb{N}))$ and*

$$ {}_MH \cong {}_M\big(p(L^2(M,\tau) \hat{\otimes} \ell^2(\mathbb{N}))\big). $$

This defines a bijection between the set of equivalence classes consisting of isomorphic separable M-modules and the set

$$ \mathcal{P}\big((M \otimes \mathrm{id}_{\ell^2(\mathbb{N})})'\big)/\sim . $$

Remark 13.14: We have to deal with the type II_∞ factor

$$ {}_MB\big(L^2(M,\tau) \hat{\otimes} \ell^2(\mathbb{N})\big) = (M \otimes \mathrm{id}_{\ell^2(\mathbb{N})})' = M' \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})) $$

where M' is of type II_1 and $B(\ell^2(\mathbb{N}))$ is of type II_∞. It carries a faithful normal semi-finite tracial weight

$$ \tau_{M'} \mathbin{\overline{\otimes}} \mathrm{Tr}\colon (M' \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})))_+ \longrightarrow [0,\infty] $$

where $\mathrm{Tr}\colon B(\ell^2(\mathbb{N}))_+ \to [0,\infty]$ is defined by $\mathrm{Tr}(x) := \sum_{n\in\mathbb{N}} \langle xe_n, e_n\rangle$ for any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$; see Exercise 4 of Sheet 5, where *Hilbert-Schmidt operators* were introduced, i. e., $x \in B(\ell^2(\mathbb{N}))$ with

$$ \|x\|_2 = \mathrm{Tr}(x^*x)^{\frac{1}{2}} < \infty. $$

Definition 13.15: Let M be a von Neumann algebra and define the set of positive elements $M_+ := \{x \in M \mid x \text{ positive}\}$ of M. A map $\mathrm{Tr}\colon M_+ \to [0,\infty]$ is called a *tracial weight*, if

- $\mathrm{Tr}(x+y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$ for all $x, y \in M_*$,
- $\mathrm{Tr}(\lambda x) = \lambda \mathrm{Tr}(x)$ for all $x \in M_+$, $\lambda \geq 0$,[1]
- $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$ for all $x \in M$.

A tracial weight $\mathrm{Tr}\colon M_+ \to [0,\infty]$ is called

(i) *semi-finite*, if for all $0 \neq x \in M_+$ there is $0 \neq y \in M_+$ such that $y \leq x$ and $\mathrm{Tr}(y) < \infty$ hold,

(ii) *normal*, if for every increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in M^+ converging strongly to $x \in M_+$ it holds $\mathrm{Tr}(x_\lambda) \to \mathrm{Tr}(x)$,

(iii) *faithful*, if for $x \in M_+$ it holds: "$\mathrm{Tr}(x) = 0 \Leftrightarrow x = 0$".

[1] We follow the convention $0 \cdot \infty := 0$.

Remark 13.16: (i) Every type II_∞ factor M is of the form $N \mathbin{\overline{\otimes}} B(\ell^2(I))$ for some type II_1 factor N and an infinite set I and thus admits a faithful normal semi-finite tracial weight $\mathrm{Tr}\colon M_+ \to [0,\infty]$, which is unique up to scaling with $\lambda > 0$.

(ii) In the situation of (i), we have

- $\mathrm{Tr}(\mathcal{P}(M)) = [0,\infty]$,
- For all $p, q \in \mathcal{P}(M)$ it holds: "$p \precsim q$ if and only if $\mathrm{Tr}(p) \leq \mathrm{Tr}(q)$".

Definition 13.17: Let M be a separable factor of type II_1. For every separable M-module (H,π), we define its *M-dimension* (or *coupling constant*) by

$$\dim_M H := (\tau_{M'} \mathbin{\overline{\otimes}} \mathrm{Tr})(vv^*) \in [0,\infty]$$

where v is chosen like in (Theorem 13.13).

Remark 13.18: Combining (Theorem 13.13), (Remark 13.14) and (Remark 13.16) (ii), we conclude that $(H,\pi) \mapsto \dim_M H$ gives a bijection between classes of equivalent separable M-modules and the set $[0,\infty]$.

Theorem 13.19: *In the situation of (Definition 13.17), we have that*

(i) $\dim_M L^2(M,\tau) = 1$,

(ii) $\dim_M(\bigoplus_{i\in I} H_i) = \sum_{i\in I} \dim_M H_i$, *if I is countable,*

(iii) $\dim_M pH = \tau_{\pi(M)'}(p) \dim_M H$ *if $\pi(M)' \subseteq B(H)$ is a type II_1 factor.*

Definition 13.20: Let $N \subseteq M$ be factors of type II_1 (with $1_M \in N$, i. e., *subfactors*). We call

$$[M : N] := \dim_N L^2(M,\tau_M)$$

the *Jones index* of N in M (note that $L^2(M,\tau_M)$ is a M-module and thus, in particular, a N-module).

Remark 13.21: Since $L^2(N,\tau_N) \subseteq L^2(M,\tau_M)$ as $\tau_M|_N = \tau_N$, we have the decomposition $L^2(M,\tau_M) = L^2(N,\tau_N) \oplus L^2(N,\tau_N)^\perp$. Thus:

$$\dim_N L^2(M,\tau_M) = \dim_N L^2(N,\tau_N) + \dim_N L^2(N,\tau_N)^\perp \geq 1 + 0 = 1,$$

i. e., $[M : N] \geq 1$ and it holds $[M : N] = 1$ if and only if $M = N$.

Example 13.22: (i) Let N be a type II_1 factor. $N \cong 1 \otimes N \subseteq M_k(\mathbb{C}) \mathbin{\overline{\otimes}} N = M$, then $[M : N] = k^2$.

(ii) Let $H \subseteq G$ be non-trivial countable discrete infinite conjugacy class groups. Then $[L(G) : L(H)] = [G : H]$.

Theorem 13.23 (Jones, 1983): *Let $N \subseteq M$ be factors of type II_1 with $[M : N] < \infty$. Then*

$$[M : N] \in [4,\infty) \cup \left\{ 4\cos^2\left(\frac{\pi}{n+2}\right) : n \in \mathbb{N} \right\}$$

and all these values show up as indices of subfactors of the hyperfinite II_1 factor $\mathcal{R}$.

14 Type III factors

At the time of von Neumann, factors of type III were more or less an enigma. This has changed totally with the work of Tomita (1967) and Takesaki (1970) on the modular theory for von Neumann algebras and the subsequent classification of type III factors by Connes (1973). We present briefly some of these pearls.

Let $M \subseteq B(H)$ be a von Neumann algebra and suppose that there exists a cyclic and separating vector $\xi \in H$. Consider the faithful normal state

$$\varphi\colon M \longrightarrow \mathbb{C}, \qquad x \longmapsto \langle x\xi, \xi\rangle.$$

Definition 14.1: We define unbounded antilinear operators on H by

$$\begin{aligned} S_0\colon H \supseteq D(S_0) &:= M\xi \longrightarrow H, \qquad x\xi \longmapsto x^*\xi \\ \text{and } F_0\colon H \supseteq D(F_0) &:= M'\xi \longrightarrow H, \qquad y\xi \longmapsto y^*\xi. \end{aligned}$$

They are well-defined since ξ is separating for M and M' (see Theorem 8.4).

Lemma 14.2: *The operators S_0 and F_0 from Definition 14.1 are densely defined. We have $F_0 \subseteq S_0^*$ and $S_0 \subseteq F_0^*$, so that S_0 and F_0 are closable.*

Proof: Since ξ is cyclic for M, we see that S_0 is densely defined. Since ξ is also separating for $M = (M')'$, it is cyclic for M' by (Theorem 8.4), thus F_0 is densely defined. Take now $x \in M$ and $y \in M'$. We have then:

$$\langle S_0(x\xi), y\xi\rangle = \langle x^*\xi, y\xi\rangle = \langle \xi, xy\xi\rangle = \langle \xi, yx\xi\rangle = \langle y^*\xi, x\xi\rangle = \langle F_0(y\xi), x\xi\rangle,$$

which shows $S_0^* \supseteq F_0$ and $F_0^* \supseteq S_0$. We define:

- S to be the closure of S_0,
- $F := S^* = S_0^*$.

(One can show that F is the closure of F_0, which gives a perfect symmetry in M and M'; this, however, is not necessary for the development of the theory).

Since $S_0 = S_0^{-1}$, it follows that S and F are injective with dense range and $S = S^{-1}$, $F = F^{-1}$. ∎

Definition 14.3: Put $\Delta := S^*S = FS$, which is densely defined, positive and invertible with $\Delta^{-1} = SF$. If $S = J\Delta^{1/2}$ is the polar decomposition (!), then the operator $J = J_\varphi\colon H \to H$ is an invertible antilinear isometry satisfying $J = J^{-1}$, $J\Delta J = \Delta^{-1}$ and $F = J\Delta^{-1/2}$. We call $\Delta = \Delta_\varphi$ the *modular operator* for (M, φ).

This generalises (Definition 13.8) to the non-tracial setting; the modular operator quantifies the failure of φ being tracial and the resulting difference between S and J.

Example 14.4: Consider the Hilbert space $H = M_n(\mathbb{C})$ with the inner product $\langle x, y\rangle = \mathrm{tr}_n(y^*x)$ and let $M = M_n(\mathbb{C}) \subseteq B(H)$ act by left-multiplication. Every positive linear functional $\varphi\colon M \to \mathbb{C}$ is of the form $\varphi(x) = \mathrm{tr}_n(h_\varphi x)$ with a unique *density matrix* $h_\varphi \geq 0$; if φ is faithful, h_φ is invertible and $\xi = h_\varphi^{1/2} \in H$ is a cyclic and separating vector with

$$\langle x\xi, \xi\rangle = \langle xh_\varphi^{1/2}, h_\varphi^{1/2}\rangle = \mathrm{tr}_n(h_\varphi x) = \varphi(x).$$

One can show that $\Delta_\varphi x = h_\varphi x h_\varphi^{-1}$ and $J_\varphi x = x^*$.

We even have an analogue of (Theorem 13.9):

Theorem 14.5 (Tomita-Takesaki theorem): *In the situation described above, we have that*

$$J_\varphi M J_\varphi = M' \quad \textit{and} \quad \Delta_\varphi^{\mathrm{i}t} M \Delta_\varphi^{-\mathrm{i}t} = M \,\forall\, t \in \mathbb{R}.$$

Note that for every $\alpha \in \mathbb{C}$, a closed operator Δ_φ^α can be defined by functional calculus. In particular, $t \mapsto \Delta_\varphi^{\mathrm{i}t}$ is a strongly continuous one-parameter group of unitaries in $B(H)$.

Definition 14.6: The strongly continuous one-parameter group

$$t \longmapsto \sigma_t^\varphi := \mathrm{Ad}\Delta_\varphi^{\mathrm{i}t}|_M, \qquad \sigma_t^\varphi(x) = \Delta_\varphi^{\mathrm{i}t} x \Delta_\varphi^{-\mathrm{i}t}$$

of automorphisms of M is called the *modular automorphism group of* (M, φ).

Proposition 14.7: *The modular automorphism group* $\{\sigma_t^\varphi \mid t \in \mathbb{R}\}$ *of* (M, φ) *satisfies the* Kubo-Martin-Schwinger boundary condition *with respect to* φ *(abbreviated KMS boundary condition), i. e.,*

(i) $\varphi = \varphi \circ \sigma_t^\varphi$ *for all* $t \in \mathbb{R}$,

(ii) *For all* $x, y \in M$, *there is a bounded continuous function* $F\colon \mathrm{cl}(\mathbb{S}) \to \mathbb{C}$ *for* $\mathbb{S} := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < 1\}$ *which is holomorphic on* $\mathbb{S}$ *such that for all* $t \in \mathbb{R}$ *it holds:*

$$F(t) = \varphi(\sigma_t^\varphi(x)y) \quad \textit{and} \quad F(t + \mathrm{i}) = \varphi(y\sigma_t^\varphi(x)).$$

(In fact, this determines σ^φ *uniquely).*

Definition 14.8: Let M be a separable factor. We put

$$S(M) := \bigcap \{\mathrm{Sp}(\Delta_\varphi) \mid \varphi\colon M \to \mathbb{C} \text{ is a faithful normal state}\}.$$

For $\mathrm{Sp}(\Delta_\varphi)$ consider $M \cong \pi_\varphi(M) \subseteq B(H_\varphi)$ for the cyclic representation $(H_\varphi, \pi_\varphi, \xi_\varphi)$; see (Theorem 13.4).

Theorem 14.9 (Connes, 1973): *Let M be a separable factor. Then:*

(i) *If $0 \in S(M)$, then M is of type III,*

(ii) *If M is of type III, then $S(M) \setminus \{0\}$ is a closed subgroup of $\mathbb{R}_+ = (0, \infty)$; thus, precisely one of the following cases occurs:*

- *M is of type III_0, i. e., $S(M) = \{0, 1\}$,*
- *M is of type III_λ, i. e., $S(M) = \{0\} \cup \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$,*
- *M is of type III_1, i. e., $S(M) = [0, \infty)$.*

Example 14.10: Take $0 < \lambda < 1$. We define $\alpha := \lambda/(1+\lambda) \in (0, \frac{1}{2})$ and

$$\varphi_\lambda \colon M_2(\mathbb{C}) \longrightarrow \mathbb{C}, \qquad \varphi_\lambda\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = \alpha a_{1,1} + (1-\alpha) a_{2,2}.$$

The *Powers factor* (Powers, 1967) given by

$$\mathcal{R}_\lambda := \overline{\bigotimes_{i \in \mathbb{N}}} (M_2(\mathbb{C}), \varphi_\lambda)$$

as an infinite tensor product, is of type III_λ. This construction is analogous to Exercise 2 of Sheet 10.

15 Universal C^*-algebras

A C^*-algebra is a Banach algebra endowed with an involution and a norm, that satisfies the C^*-condition. This axiomatic definition allows for a universal construction:

Construction 15.1: Let $E = \{x_i \mid i \in I\}$ be a set of *generators*, I an index set. Let $P(E)$ be the involutive $\mathbb{C}$-algebra of non-commutative polynomials in $E \cup E^*$, where $E^* := \{x_i^* \mid i \in I\}$, $(\lambda x_{i_1} \cdots x_{i_k})^* := \overline{\lambda} x_{i_k}^* \cdots x_{i_1}^*$. Hence

$$P(E) = \langle \{x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \mid \alpha_i \in \{1, {}^*\}\} \rangle.$$

Let $R \subseteq P(E)$ be a set of *relations*. Let $J(R) \subseteq P(E)$ be the two-sided ideal in $P(E)$ generated by R. Put

$$A(E,R) := P(E)/J(R),$$

the *universal involutive algebra with generators E and relations R*. For $x \in A(E,R)$, put

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E,R)\}.$$

Here p is a C^*-seminorm if and only if $p(\lambda x) = |\lambda| p(x)$, $p(x+y) \leq p(x) + p(y)$, $p(xy) \leq p(x)p(y)$ and $p(x^*x) = p(x)^2$ for all $x, y \in A(E,R)$, $\lambda \in \mathbb{C}$. If now $\|x\| < \infty$ for all $x \in A(E,R)$, put

$$C^*(E|R) := \mathrm{cl}_{\|\cdot\|}(A(E,R)/\{x \in A(E,R) \mid \|x\| = 0\}),$$

the *universal C^*-algebra with generators E and relations R.*

Remark 15.2: (i) The principle used in Construction 15.1 is more or less the same principle as in "$\{$non-commutative polynomials in x and $1\} = C^*(x,1) \subseteq A$".

(ii) "Relations" really means: If $x_1^* x_1 - 1 \in R$, then $x_1^* x_1 = 1$ in $C^*(E|R)$.

(iii) If A is any given *-algebra, $C^*(A) := \mathrm{cl}_{\|\cdot\|}(A/\{x \mid \|x\| = 0\})$, where $\|x\| := \sup\{p(x) \mid p \text{ is } C^*\text{-seminorm on } A\}$; the so called *enveloping C^*-algebra of A*, if $\|x\| < \infty$ for all $x \in A(E,R)$.

Proposition 15.3: *The universal C^*-algebras have the following* universal property*: Let B be a C^*-algebra, $E' := \{y_i \in B \mid i \in I\} \subseteq B$ be a subset satisfying the relations R. Then there is a unique *-homomorphism $\varphi\colon C^*(E|R) \to B$ with $\varphi(x_i) = y_i$.*

Proof: Consider the following diagram:

$$\begin{tikzcd}
A(E,R) \arrow[rr, "\varphi_0"] \arrow[dr, "\subseteq"'] & & B \\
& C^*(E|R) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

There is a so called *replacement homomorphism* φ_0, since $\varphi_0(R) = 0$ and this replacement homomorphism is continuous (by the definition of $\|x\|$; $p(x) := \|\varphi_0(x)\|_B$ defines a C^*-seminorm and it holds $\|\varphi_0(x)\| \leq \|x\|$), thus extends to a *-homomorphism $\varphi\colon C^*(E|R) \to B$. ■

Lemma 15.4: *If there is a constant $C > 0$ such that $p(x_i) < C$ for all C^*-seminorms p on $A(E,R)$ and all $i \in I$, then $C^*(E|R)$ exists (i. e., it holds $\|x\| < \infty$ for all $x \in A(E,R)$).*

Proof: Let $x = x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ ($\alpha_i \in \{1, {}^*\}$) be a monomial and $p(x) \leq C^k$. Then this also holds for polynomials. ■

Proposition 15.5: *Let $\mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then*

$$C(\mathbb{S}^1) \cong C^*(u, 1 \mid u^*u = uu^* = 1).$$

Hence, $C(\mathbb{S}^1)$ is the universal C^*-algebra generated by a unitary.

Proof: $C^*(u,1) = C^*(u,1|\dots)$ has $E = \{1, u\}$ and the relations $u^*u - 1$, $uu^* - 1$, $1u - u$, $u1 - u$, $11 - 1$, etc. It exists, since $p(1)^2 = p(1^*1) = p(1) \leq 1$ and $p(u)^2 = p(u^*u) = p(1) \leq 1$.

The identity function defined by $z(t) = t$ for $t \in \mathbb{S}^1$ and $1(t) \equiv 1$ for $t \in \mathbb{S}^1$ satisfy $z^*z = zz^* = 1$. Hence, by the universal property, there is *-homomorphism

$$\begin{aligned} \varphi_0\colon C^*(u,1) &\longrightarrow C(\mathbb{S}^1),\\ u &\longmapsto z,\\ 1 &\longmapsto 1. \end{aligned}$$

On the other hand $C^*(u,1) \cong C(\mathrm{Sp}(u))$ by the theorem of Gelfand-Naimark (or rather the functional calculus) and $\mathrm{Sp}(u) \subseteq \mathbb{S}^1$. Hence we have the diagram

$$\begin{tikzcd} C^*(u,1) \arrow[rr, "\varphi"] \arrow[dr, "\cong"'] & & C(\mathbb{S}^1) \arrow[ll, bend left, "\psi"] \arrow[dl, "f|_{\mathrm{Sp}(u)} \mapsfrom f"] \\ & C(\mathrm{Sp}(u)) & \end{tikzcd}$$

It holds by functional calculus

$$\varphi \circ \varphi(u) = \psi(z) = u,$$

hence φ is injective; in total we conclude that φ is an isomorphism.

The intuition is: A “universal unitary u” should have the maximal spectrum, so $\mathrm{Sp}(u) = \mathbb{S}^1$. Hence $C^*(u,1) \cong C(\mathrm{Sp}(u)) = C(\mathbb{S}^1)$. ■

Example 15.6: The universal C^*-algebra $C^*(x|x = x^*)$ does *not* exist! There are elements $y = y^* \in B$ with arbitrarily large norm, hence $p(x) = \|y\|$ can be large, hence $\|x\| = \infty$ for some $x \in A(x, x = x^*)$.

Proposition 15.7: *For $n \geq 2$, the following are isomorphic:*

(i) $M_n(\mathbb{C})$,

(ii) $C^*(e_{i,j}, i, i = 1, \ldots, n \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \, \forall \, i, j, k, l)$,

(iii) $C^*(x_1, \ldots, x_n \mid x_i^* x_j = \delta_{i,j} x_1)$.

Proof: "$M_n(\mathbb{C}) \cong C^*(e_{i,j}, i, i = 1, \ldots, n \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \, \forall \, i, j, k, l)$": Put $E_{i,j} := (\delta_{i,k}\delta_{j,l})_{1 \leq k,l \leq n} \in M_n(\mathbb{C})$. These $E_{i,j}$ satisfy the relations stated in the universal C^*-algebra in (ii), thus there is

$$\begin{aligned} \varphi \colon C^*(e_{i,j}, i, i = 1, \ldots, n \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \, \forall \, i, j, k, l) &\longrightarrow M_n(\mathbb{C}), \\ e_{i,j} &\longmapsto E_{i,j} \end{aligned}$$

by the universal property, which is surjective. Since the universal C^* algebra $C^*(e_{i,j}, i, i = 1, \ldots, n \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \, \forall \, i, j, k, l)$ is n^2-dimensional, and so is $M_n(\mathbb{C})$, φ is injective.

Note that $C^*(e_{i,j}, i, i = 1, \ldots, n \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l} \, \forall \, i, j, k, l)$ exists, since $p(e_{i,j})^2 = p(e_{i,j}^* e_{i,j}) = p(e_{j,j})$, $p(e_{j,j})^2 = p(e_{j,j}) \in \{0, 1\}$.

The other isomorphy is an exercise on Sheet 12:

$$\begin{tikzcd} C^*(e_{i,j}) \arrow[r, bend right, "\varphi"'] & C^*(x_1, \ldots, x_n) \arrow[l, bend right, "\psi"'] \end{tikzcd}$$

such that $\varphi \circ \psi = \mathrm{id}_{C^*(x_1, \ldots, x_n | \ldots)}$ and $\psi \circ \varphi = \mathrm{id}_{C^*(e_{i,j}, 1 \leq i,j \leq n | \ldots)}$. ■

Remark 15.8: On Sheet $11 \pm \varepsilon$ in Functional Analysis I, we proved that $M_n(\mathbb{C})$ is simple. Hence, if B is a C^*-algebra with $y_1, \ldots, y_n \in B$ and $y_i^* y_j = \delta_{i,j} y_1$, then $M_n(\mathbb{C}) \cong C^*(y_1, \ldots, y_n) \subseteq B$, because for

$$\varphi \colon M_n(\mathbb{C}) \cong C^*(x_1, \ldots, x_n) \longrightarrow C^*(y_1, \ldots, y_n) \subseteq B$$

we know that $\ker(\varphi) \triangleleft M_n(\mathbb{C})$; hence $\ker(\varphi) = \{0\}$.

Proposition 15.9: *The following C^*-algebras are isomorphic:*

(i) *The compact operators $K(H)$ on a separable Hilbert space H,*

(ii) $C^*(e_{i,j}, i, j \in \mathbb{N} \mid e_{i,j}^* = e_{j,i}, e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l})$,

(iii) $C^*(x_i, i \in \mathbb{N} \mid x_i^* x_j = \delta_{i,j} x_1)$

In (ii) and (iii) we may replace $\mathbb{N}$ by an arbitrary infinite countable set.

Proof: "(i) $\cong$ (ii)": Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Define the operators $f_{i,j}(e_n) := \delta_{j,n} e_i$. Then $f_{i,j} \in K(H)$ since it is of finite rank. By the universal property, we find

$$\varphi\colon C^*(e_{i,j}, i,j \in \mathbb{N} \mid \dots) \longrightarrow K(H), \qquad e_{i,j} \longmapsto f_{i,j},$$

furthermore φ is surjective, since the finite rank operators are dense in $K(H)$. It remains to be shown that φ is injective. Put therefore

$$M_n := C^*(e_{i,j} \mid 1 \leq i,j \leq n) \subseteq C^*(e_{i,j}, i,j \in \mathbb{N} \mid \dots).$$

By (Proposition 15.7) and (Remark 15.8), we have $M_n \cong M_n(\mathbb{C})$. This shows that $\varphi|_{M_n} \to K(H)$ is injective, since $\ker(\varphi|_{M_n}) \triangleleft M_n \cong M_n(\mathbb{C})$; thus $\varphi|_{M_n}$ is isometric. Now as φ is isometric on the dense subset $\bigcup_{n\in\mathbb{N}} M_n \subseteq C^*(e_{i,j} i,j \in \mathbb{N} \mid \dots)$, φ is isometric on $C^*(e_{i,j}, i,j \in \mathbb{N} \mid \dots)$.

"(ii) $\cong$ (iii)": Exercise on Sheet 12. ■

Remark 15.10: $K(H)$ is simple (this is an exercise on Sheet 12), so the analog of (Remark 15.8) holds true.

16 Example: Toeplitz algebra $\mathcal{T}$

Definition 16.1: We denote by $\mathcal{T} := C^*(v, 1 \mid v^*v = 1)$ the universal C^*-algebra generated by an isometry, the so called *Toeplitz-algebra*.

Theorem 16.2: (i) *The ideal $\langle 1 - vv^* \rangle$ in $\mathcal{T}$ (the ideal generated by $1 - vv^*$) is isomorphic to $K(H)$ for some separable Hilbert space H and we have*

$$\mathcal{T}/\langle 1 - vv^* \rangle \cong C^*(\mathbb{S}^1).$$

In other words, the sequence $0 \to K(H) \to \mathcal{T} \to C(\mathbb{S}^1) \to 0$ is exact.

(ii) *The map*

$$\varphi \colon \mathcal{T} \longrightarrow C^*(S) \subseteq B(\ell^2(\mathbb{N})), \qquad v \longmapsto S,$$

where S denotes the unilateral shift (defined by $Se_n = Se_{n+1}$), is an isomorphism.

Proof: (i) We want to prove the assertion in small steps:

(1) $1 - vv^* \in \mathcal{T}$ is a projection.

(2) $\langle 1 - vv^* \rangle = \mathrm{cl}(\mathrm{span}\{v^k(1 - vv^*)v^{*l} \mid k, l \in \mathbb{N}\}) =: A \subseteq \mathcal{T}$.

Proof (of (2)): "$\supseteq$" is clear. "$\subseteq$": By construction, A is a closed vector space. It holds $vx, v^*x \in A$ for all $x \in A$ since $v^*(1 - vv^*) = 0$, thus $yx, xy \in A$ for all $y \in T$ and $x \in A$, i. e., A is an ideal. ■

(3) $f_{k,l} := v^k(1 - vv^*)v^{*l}$ satisfy the relations of $K(H) = C^*(e_{i,j}, i, j \in \mathbb{N}_0 \mid \dots)$, as we have the following:

$$f_{i,j}f_{k,l} = v^i(1 - vv^*)v^{*j}v^k(1 - vv^*)v^{*l} = \delta_{j,k}v^i(1 - vv^*)v^{*l}.$$

Thus there is an injective map

$$\eta \colon K(H) \hookrightarrow \mathcal{T}, \qquad e_{i,j} \longmapsto f_{i,j}$$

with $\eta(K(H)) = \langle 1 - vv^* \rangle$ by (2).

(4) We have the following diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & C(\mathbb{S}^1) \arrow[dl, "\alpha"] \\ & \mathcal{T}/\langle 1 - vv^* \rangle & \end{tikzcd}$$

But $\psi(\langle 1 - vv^* \rangle) = 0$, hence there is a map

$$\beta \colon \mathcal{T}/\langle 1 - vv^* \rangle \longrightarrow C(\mathbb{S}^1), \qquad \pi(v) \longmapsto u$$

thus $\alpha \circ \beta = \mathrm{id}$, $\beta \circ \alpha = \mathrm{id}$ thus $C(\mathbb{S}^1) \cong \mathcal{T}/\langle 1 - vv^* \rangle$.

(ii) As for part (i), we want to show the assertion in small steps:

(1) φ exists (since S is an isometry) and is surjective. We need to show that $\|\varphi(x)\| = \|x\|$, but $\|x\| = \sup\{\|\rho(x)\| \mid \rho \text{ irreducible representation of } \mathcal{T}\}$ (refer to Remark 5.18 (iv)), hence we need to show $\|\rho(x)\| \leq \|\varphi(x)\|$ for all irreducible representations $\rho\colon \mathcal{T} \to B(H)$ (of course it holds $\|\varphi(x)\| \leq \|x\|$).

(2) *Claim:* Let $\rho\colon \mathcal{T} \to B(H)$ be irreducible, $\rho(p) \neq 0$ with $p := 1 - vv^*$. Then ρ and φ are unitarily equivalent and then $\|\rho(x)\| = \|U\varphi(x)U^*\| \leq \|\varphi(x)\|$.

Proof (of (2)): We have: For all $x \in \mathcal{T}$ there is $\lambda \in \mathbb{C}$ such that $pxp = \lambda p$ (because if $x = v^k v^{*l}$ is a monomial with $k, l \in \mathbb{N}_0$, then

$$pxp = (1 - vv^*)v^k v^{*l}(1 - vv^*) = \delta_{k,0}\delta_{l,0}p$$

which transfers to polynomials and limits. Thus there is a state $f\colon \mathcal{T} \to \mathbb{C}, x \mapsto \lambda$ with $pxp = f(x)p$ for all $x \in \mathcal{T}$. Then for all $x \in \mathcal{T}$ we have

$$\langle \pi_f(x)\xi_f, \xi_f\rangle = f(x) = \langle \varphi(x)e_1, e_1\rangle,$$

as $\langle \varphi(x)e_1, e_1\rangle = \langle S^k S^{*l} e_1, e_1\rangle = \delta_{0,l}\delta_{0,k} = f(x)$, if x is a monomial, which again transfers to polynomials and limits. Therefore, π_f and φ are unitarily equivalent. Since $\rho(p) \neq 0$, we find $\xi \in H_\rho$ such that $\|\xi\| = 1$ and $\rho(p)\xi = \xi$. Then

$$\langle \rho(x)\xi, \xi\rangle = \langle \rho(x)\rho(p)\xi, \rho(p)\xi\rangle = \langle \rho(pxp)\xi, \xi\rangle = f(x)\langle \rho(p)\xi, \xi\rangle = f(x),$$

hence also ρ is unitarily equivalent to f. ∎

(3) Let $\rho\colon \mathcal{T} \to B(H)$ be irreducible $\rho(p) = 0$. Then $\rho(v)$ is a unitary. Moreover, $\langle 1 - SS^*\rangle \lhd C^*(S)$ is isomorphic to $K(H)$. Hence $\pi\colon C^*(S) \to C^*(S)/\langle 1 - SS^*\rangle$ satisfies that $\pi(S)$ is unitary. It remains to be shown that $\mathrm{Sp}(\pi(S)) = \mathbb{S}^1$, as then we have the following diagram

$$\begin{tikzcd}
 & \mathcal{T} \arrow[r, "\rho"] \arrow[dl, "\varphi"'] & C^*(\rho(v)) \subseteq B(H) & \\
C^*(S) \arrow[r, "\pi"'] & C^*(S)/\langle 1 - SS^*\rangle \arrow[r, "\cong"] & C(\mathrm{Sp}(\pi(S))) \arrow[r, "="] & C(\mathbb{S}^1) \arrow[ul, "\alpha"']
\end{tikzcd}$$

and thus $\|\rho(x)\| \leq \|\alpha \circ \pi \circ \varphi(x)\| \leq \|\varphi(x)\|$.

Proof: Let $\lambda \in \mathbb{S}^1$. Consider $d(\lambda)e_n := \lambda^n e_n$, then $d(\lambda) \in B(H)$ is unitary with $d(\lambda)^* = d(\overline{\lambda})$. We have $d(\lambda)Sd(\lambda)^* = \lambda S$ thus

$$\beta\colon C^*(S) \longrightarrow C^*(S), \qquad x \longmapsto d(\lambda)xd(\lambda)^*$$

is an automorphism with $\beta(\langle 1 - SS^*\rangle) \subseteq \langle 1 - SS^*\rangle$ and $\beta(S) = \lambda S$. Hence we have the map

$$\dot{\beta}\colon C^*(S)/\langle 1 - SS^*\rangle \xrightarrow{\cong} C^*(S)/\langle 1 - SS^*\rangle, \qquad \dot{\beta}(\pi(S)) = \lambda\pi(S), \qquad ∎$$

which shows $\mathrm{Sp}(\pi(S)) = \mathrm{Sp}(\dot{\beta}(\pi(S))) = \lambda\mathrm{Sp}(\pi(S))$ for all $\lambda \in \mathbb{S}^1$, thus finally $\mathrm{Sp}(\pi(S)) = \mathbb{S}^1$. ∎

Remark 16.3: $L^2(\mathbb{S}^1)$ has the orthonormal basis $e^n := (z \mapsto z^n)$ for $n \in \mathbb{Z}$. One can show that

$$\mathcal{T} \longrightarrow \{T_f + k \mid f \in C(\mathbb{S}^1), k \in K(L^2(\mathbb{S}^1))\}, \qquad v \longmapsto T_z,$$

where $T_f := P_{H^2} M_f$, $M_f(g) := fg$ for $g \in L^2(\mathbb{S}^1)$ and $H^2 := \operatorname{span}\{e_n \mid n \geq 0\} \subseteq L^2(\mathbb{S}^1)$. Oftentimes, $\mathcal{T}$ is introduced in this way. $\{T_f + k \mid f \in C(\mathbb{S}^1), k \in K(L^2(\mathbb{S}^1))\}$ are called the Toeplitz operators.

17 The (irrational) rotation algebra A_ϑ

Definition 17.1: Let $\vartheta \in \mathbb{R}$. Put

$$A_\vartheta := C^*(u, v \text{ unitaries} \mid uv = e^{2\pi i \vartheta} vu)$$

and call A_ϑ *rotation algebra.* For $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$, we call A_ϑ irrational rotation algebra.

Let in the following $\lambda := e^{2\pi i \vartheta}$.

Remark 17.2: (i) A_ϑ exists, since u, v are unitaries.

(ii) $A_{\vartheta=0} \cong C(\mathbb{T}^2)$, where $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ is the 2-torus, via

$$u \longmapsto \big((z_1, z_2) \mapsto z_1\big), \qquad v \longmapsto \big((z_1, z_2) \mapsto z_2\big).$$

The existence of this *-homomorphism is granted by the universal property and it is an isomorphism since $A_{\vartheta=0} \cong C(\mathrm{Spec}(A_{\vartheta=0})) \cong C(\mathbb{T}^2)$ where the first isomorphy is ensured by the first fundamental theorem of C^*-algebras and the second isomorphy holds, because $\varphi \in \mathrm{Spec}(A_{\vartheta=0})$ is determined by the values of $\varphi(u) \in \mathbb{S}^1$ and $\varphi(v) \in \mathbb{S}^1$.

Hence, for $\vartheta \neq 0$, we may view A_ϑ as a non-commutative function algebra on a "non-commutative torus $\mathbb{T}^2_\vartheta$".

(iii) $\vartheta \notin \mathbb{Q}$ behave very differently from $\vartheta \in \mathbb{Q}$; that's why mostly the irrational rotation algebras are studied (see Sheet 13 and Theorem 17.xx).

(iv) The non-commutative tori are examples of non-commutative manifolds in A. Connes "non-commutative geometry", an analog of differential geometry. Such a non-commutative differential geometry is supposed to be relevant for quantum physics ("At a deep and perhaps fundamental level, quantum field theory and non-commutative geometry are made of the same stuff.", Gracia-Bondia et. al, page 522), for instance for the quantum Hall effect.

Proposition 17.3: *A_ϑ has some concrete representations (i. e., $A_\vartheta \neq 0$).*

(i) *Let $\tilde{S} \in B(\ell^2(\mathbb{Z}))$ be the bilateral shift and consider the map d defined by $d(\lambda)e_n := \lambda^n e_n$. We have*

$$\pi \colon A_\vartheta \longrightarrow B(\ell^2(\mathbb{Z}), \qquad u \longmapsto d(\lambda), v \longmapsto \tilde{S}.$$

(ii) *Let $\tilde{u}$, $\tilde{v}$ be the following operators:*

$$\begin{aligned} \tilde{u} \colon L^2(\mathbb{S}^1) \longrightarrow L^2(\mathbb{S}^1), &\qquad (\tilde{u}f)(t) := f(\lambda t), \\ \tilde{v} \colon L^2(\mathbb{S}^1) \longrightarrow L^2(\mathbb{S}^1), &\qquad (\tilde{v}f)(t) := tf(t). \end{aligned}$$

We have

$$\pi \colon A_\vartheta \longrightarrow B(L^2(\mathbb{S}^1)), \qquad u \longmapsto \tilde{u}, v \longmapsto \tilde{v}.$$

Proof: (i) $\tilde{S}$ is unitary and as $d(\lambda)^* = d(\overline{\lambda})$, $d(\lambda)$ is unitary as well. Now it holds

$$d(\lambda)\tilde{S}e_n = d(\lambda)e_{n+1} = \lambda^{n+1}e_{n+1} = \lambda\tilde{S}d(\lambda)e_n,$$

hence π exists.

(ii) Apply $\tilde{u}\tilde{v}$ to $e_n := (z \mapsto z^n)$. ■

We ask: Are the representations in Proposition 17.3 isomorphic? Does

$$C^*(\tilde{S}, d(\lambda)) \cong C^*(\tilde{u}, \tilde{v})$$

hold? Are the maps π in Proposition 17.3 injective? The answer to all these questions is "yes (if $\vartheta \notin \mathbb{Q}$)", and the reason for this is that A_ϑ is simple (for $\vartheta \notin \mathbb{Q}$), which we now want to show.

Definition 17.4: Let A be a unital C^*-algebra.

(i) If $B \subseteq A$ is a C^*-algebra such that $1_A \in B$ and $\varphi\colon A \to B$ is positive, linear, unital and $\varphi^2 = \varphi$, we call φ a *(conditional) expectation*. φ is called faithful, if for all $a \geq 0$ with $\varphi(a) = 0$ it holds $a = 0$.

(ii) $\tau\colon A \to \mathbb{C}$ is a (normalised) trace, if τ is positive, linear and it holds $\tau(1) = 1$, $\tau(ab) = \tau(ba)$ for all $a, b \in A$. τ is faithful, if for all $a \geq 0$ with $\varphi(a) = 0$ it holds $a = 0$.

Proposition 17.5: *Let $\vartheta \notin \mathbb{Q}$.*

(i) *For $\xi, \mu \in \mathbb{S}^1$, the map*

$$\rho_{\xi,\mu}\colon A_\vartheta \longrightarrow A_\vartheta, \qquad u \longmapsto \xi u, v \longmapsto \mu v$$

is an automorphism (i. e., a *-isomorphism).

(ii) *The maps $\varphi_1, \varphi_2\colon A_\vartheta \to A_\vartheta$ defined via*

$$\varphi_1(x) := \int_0^1 \rho_{1,e^{2\pi \mathrm{i} t}}(x)\,dt, \qquad \varphi_2 := \int_0^1 \rho_{e^{2\pi \mathrm{i} t},1}(x)\,dt$$

are contractive (i. e., $\|\varphi_i\| \leq 1$), faithful expectations.

(iii) *We have $\varphi_1(A_\vartheta) \subseteq C^*(u) \subseteq A_\vartheta$, $\varphi_2(A_\vartheta) = C^*(v) \subseteq A_\vartheta$, $\varphi_1|_{C^*(u)} = \mathrm{id}_{C^*(u)}$, $\varphi_2|_{C^*(v)} = \mathrm{id}_{C^*(v)}$,*

(iv) *We have for any $a_{k,l} \in \mathbb{C}$ (where only finitely many $a_{k,l} \neq 0$)*

$$\varphi_1\Big(\sum_{k,l\in\mathbb{Z}} a_{k,l}u^k v^l\Big) = \sum_{k\in\mathbb{Z}} a_{k,0}u^k, \qquad \varphi_2\Big(\sum_{k,l\in\mathbb{Z}} a_{k,l}u^k v^l\Big) = \sum_{l\in\mathbb{Z}} a_{0,l}v^l$$

(v) *For $\vartheta \notin \mathbb{Q}$ it holds*

$$\varphi_1(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x u^{-j}, \qquad \varphi_2(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} v^j x v^{-j}.$$

This statement is false for $\vartheta \in \mathbb{Q}$!

Proof: (i) $\rho_{\xi,\mu}$ exists by the universal property and because $\rho_{\overline{\xi},\overline{\mu}} \circ \rho_{\xi,\mu} = \mathrm{id}$, $\rho_{\xi,\mu}$ is an automorphism.

(ii) Elements in A_ϑ may be approximated by $\sum_{k,l\in\mathbb{Z}} a_{k,l} u^k v^l$ with $a_{k,l} \neq 0$ finitely often. For $x \in A_\vartheta$, the map

$$f_x \colon \mathbb{T}^2 \longrightarrow A_\vartheta, \qquad (\xi,\mu) \longmapsto \rho_{\xi,\mu}(x)$$

is norm continuous, since for $x = \sum_{k,l=-n}^{n} a_{k,l} u^k v^l$ it holds

$$\|f_x(\xi_1,\mu_1) - f_x(\xi_2,\mu_2)\| \leq \sum_{k,l=-n}^{n} |\xi_1^k \mu_1^l - \xi_2^k \mu_2^l| |a_{k,l}| \to 0 \text{ for } (\xi_1,\mu_1) \to (\xi_2,\mu_2),$$

thus

$$g_x \colon [0,1] \longrightarrow A_\vartheta, \qquad t \longmapsto f_x(1, e^{2\pi \mathrm{i} t})$$

is also norm-continuous, hence $\frac{1}{n}\sum_{j=1}^{n} g_x(t_j) \to \int_0^1 g_x(t)\,dt =: \varphi_1(x)$ exists as a limit of Riemann sums (for $0 \leq t_0 \leq \cdots \leq t_n = 1$).

To see that φ_1 is contractive, we notice that

$$\|\varphi_1(x)\| \leq \frac{1}{n} \Big\| \sum_j \rho_{1,e^{2\pi \mathrm{i} t_j}}(x) \Big\| \leq \frac{1}{n} \sum_j \|x\| = \|x\|.$$

φ_1 is positive because for $x \geq 0$, $\rho_{1,e^{2\pi \mathrm{i} t}}(x) \geq 0$, hence $\varphi_1(x)$ is positive as limit of positive elements. One checks, that φ_1 is linear, unital and faithful.

Now, by (iii), $\varphi_1(\varphi_1(x)) = \mathrm{id}|_{C^*(u)}(\varphi_1(x)) = \varphi_1(x)$, as $\varphi_1(x) \in C^*(u)$.

(iii) It holds

$$\varphi_1(v^l) = \int_0^1 \rho_{1,e^{2\pi \mathrm{i} t}}(v^l)\,dt = \int_0^1 e^{2\pi \mathrm{i} l t} v^l\,dt = \left(\int_0^1 e^{2\pi \mathrm{i} l t}\,dt\right) v^l = \delta_{0,l},$$

hence $\varphi_1(u^k v^l) = \int_0^1 \rho_{1,e^{2\pi \mathrm{i} t}}(u^k v^l)\,dt = u^k \varphi_1(v^l) = \delta_{0,l} u^k$, so $\varphi_1(A_\vartheta) \subseteq C^*(u)$ and $\varphi_1|_{C^*(u)} = \mathrm{id}_{C^*(u)}$.

(iv) This was also shown in (iii).

(v) We compute

$$\frac{1}{2n+1} \sum_{j=-n}^{n} u^j (u^k v^l) u^{-j} = \frac{1}{2n+1} \Big(\sum_{j=-n}^{n} \lambda^{jl} \Big) u^k v^l \to \delta_{0,l} u^k v^l = \varphi_1(u^k v^l).$$

■

Corollary 17.6: *Let $\vartheta \notin \mathbb{Q}$. The map*

$$\tau := \varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 \colon A_\vartheta \longrightarrow \mathbb{C}$$

is the unique unital faithful trace on A_ϑ. We have $\tau(\sum_{k,l\in\mathbb{Z}} a_{k,l} u^k v^l) = a_{0,0}$, this is "the $(0,0)$-th Fourier-coefficient".

Proof: It holds for $k, l \in \mathbb{Z}$:

$$\varphi_1\varphi_2(u^k v^l) = \delta_{k,0}\varphi_1(v^l) = \delta_{k,0}\delta_{l,0} = \varphi_2\varphi_1(u^k v^l),$$

hence τ is well-defined. Furthermore τ is linear, positive, unital and faithful as a composition of two maps with these properties, also

$$\begin{aligned} \tau((u^k v^l)(u^m v^n)) = \lambda^{-lm}\tau(u^{m+k}v^{l+n}) &= \delta_{m+k,0}\delta_{l+n,0}\lambda^{-lm} \\ &= \lambda^{-}nk\tau(u^{m+k}v^{l+n}) = \tau((u^m u^n)(u^k v^l)). \end{aligned}$$

One can check, that τ is unique. ■

Theorem 17.7: *For $\vartheta \notin \mathbb{Q}$, A_ϑ is simple.*

Theorem 17.7 is false for $\vartheta \in \mathbb{Q}$, see Exercise sheet 13.

Proof: Let $0 \neq I \triangleleft A_\vartheta$, hence we find $0 \neq x \in I$, i.e., $0 \neq x^*x \in I$. Then

$$0 \neq \varphi_1(x^*x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} u^j x^* x u^{-j} \in I,$$

but $0 \neq \tau(x^*x) = \varphi_2\varphi_1(x^*x) \in I$, thus $1 \in I$, i.e., $I = A_\vartheta$ (because $\tau(x^*x) \in \mathbb{C}$ and $\varphi_1(x^*x) \in I$). ■

Remark 17.8: It holds $\vartheta' = \pm\vartheta \pmod{\mathbb{Z}}$ if and only if $A_\vartheta \cong A_{\vartheta'}$ ("$\Rightarrow$" is trivial, "$\Leftarrow$" requires certain Powers-Rieffel projections).

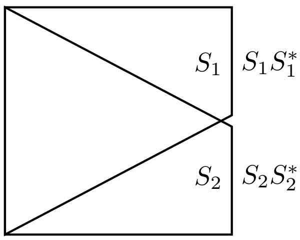
18 Cuntz-Algebra

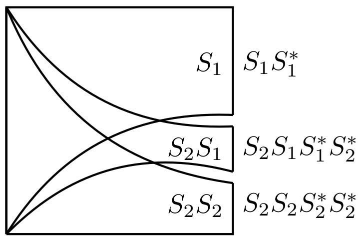
Definition 18.1: The *Cuntz-Algebra* $\mathcal{O}_n$ for $2 \leq n < \infty$ is defined as

$$\mathcal{O}_n := C^*(S_1, \dots, S_n \text{ isometries} \mid \textstyle\sum_{i=1}^n S_i S_i^* = 1),$$

for $n = \infty$ put $\mathcal{O}_\infty := C^*(S_1, S_n, \dots \text{ projections} \mid S_i^* S_j = \delta_{i,j})$.

Remark 18.2: (i) $\mathcal{O}_n$ is the structure of decomposing a space into n copies.





(ii) J. Cuntz introduced these C^*-algebras in 1977. They are important (counter-) examples for certain questions. But they are also building blocks of the theory of C^*-algebras via theorems like:

- Let A be a separable C^*-Algebra. A is exact if and only if there is an embedding $A \hookrightarrow \mathcal{O}_2$.
- Let A be a C^*-algebra. A is a unital, simple, separable, nuclear C^*-algebra if and only if $A \cong A \otimes \mathcal{O}_2$.
- Let A be a C^*-algebra. A is purely infinte if and only if $A \cong A \otimes \mathcal{O}_\infty$.

The above stated theorems are called the Kirchberg-Phillips theorems.

(iii) Also, it has nice properties, like being purely infinite (a strengthening of being simple). We will prove this property in this chapter.

Definition 18.3: A *word* in $\mathcal{O}_n$ is $S_\mu := S_{i_1} \cdots S_{i_k}$, where $\mu = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ is a multi-index. We call $|\mu| = k$ the *length of the multi-index* μ.

Lemma 18.4: (i) *It holds* $S_i^* S_j = \delta_{i,j}$,

(ii) *Let* μ, ν *be multi-indices with* $|\mu| = |\nu|$. *Then* $S_\mu^* S_\nu = \delta_{\mu,\nu}$, *where* $\delta_{\mu,\nu}$ *is defined standing to reason.*

(iii) *Let* μ, ν *be multi-indices. Then we have the following:*

(1) *If* $|\mu| < |\nu|$, *then*

$$S_\mu^* S_\nu = \begin{cases} S_{nu'}, & \text{if } \nu = \mu\nu', \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If* $|\mu| > |\nu|$, *then*

$$S_\mu^* S_\nu = \begin{cases} S_{\mu'}^*, & \text{if } \mu = \nu\mu', \\ 0, & \text{otherwise.} \end{cases}$$

Proof: (i) The case $i = j$ is clear: $S_i^* S_i = 1$, because S_i are isometries.
For $i \neq j$, we have $S_i S_i^* + S_j S_j^* \leq \sum_k S_k S_k^* = 1$, thus $S_i^* S_j S_j^* S_i \leq 0$. Hence

$$S_i^*(S_i S_i^* + S_j S_j^*) S_i \leq S_i^* S_i = S_i^*(1 + S_i^* S_j S_j^* S_i) S_i = 1$$

but as $S_i^* S_j S_j^* S_i = (S_j^* S_i)^*(S_j^* S_i) \geq 0$, we get that $(S_j^* S_i)^*(S_j^* S_i) = 0$, so finally $S_j^* S_i = 0$.

(ii) Exercise on Sheet 13,

(iii) Exercise on Sheet 13. ■

Proposition 18.5: (i) *For* $k \in \mathbb{N}$ *define*

$$\mathcal{F}_k^n := \operatorname{span}\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k\} \subseteq \mathcal{O}_n.$$

Note that $\mathcal{F}_k^n \cong M_{n^k}(\mathbb{C})$, *here* $S_1^k S_1^{k*} \leftrightarrow e_{1,1}$.

(ii) *The set* $\{S_\mu S_\nu^* \mid \mu, \nu \text{ arbitary}\} \subseteq \mathcal{O}_n$ *is dense.*

Proof: (i) Put $e_{\mu\nu} := S_\mu S_\nu^* \in \mathcal{F}_k^n$. Then we have the relations $e_{\mu\nu}^* = e_{\nu\mu}$ as well as $e_{\mu\nu} e_{\rho\sigma} = S_\mu S_\nu^* S_\rho S_\sigma^* = \delta_{\nu\rho} e_{\mu\sigma}$. Due to $|\{\mu \in \{1, \ldots, n\}^k\}| = n^k$, we obtain that

$$\varphi\colon M_{n^k}(\mathbb{C}) \longrightarrow \mathcal{F}_n^k, \qquad e_{\mu\nu} \longmapsto S_\mu S_\nu^*$$

is a *-isomorphism.

(ii) Monomials in $\mathcal{O}_n$ are of the form $S_\mu S_\nu^*$ by (Lemma 18.4). ■

Lemma 18.6: (i) *Let* $k \in \mathbb{N}$. *Then*

$$\sum_{\substack{\delta \text{ multi-index}, \\ |\delta| = k}} S_\delta S_\delta^* = 1.$$

(ii) *For* $l \leq k$, *we have* $\mathcal{F}_l^n \subseteq \mathcal{F}_k^n$.

Proof: (i) This is exercise 2b) on sheet 13.

(ii) Let $S_\mu S_\nu^* \in \mathcal{F}_l^n$ where $l \leq k$ and $|\mu| = |\nu| = l$. Then by (i) it holds

$$\sum_{\substack{\delta \text{ multi-index}, \\ |\delta| = k-l}} S_\nu S_\delta S_\delta^* S_\nu^* \in \mathcal{F}_k^n.$$

■

Lemma 18.7: *Let μ, ν be multi-indices with $|\mu|, |\nu| \leq k$, $|\mu| \neq |\nu|$. Then, for $S_\gamma := S_1^k S_2$,*

$$S_\gamma^*(S_\mu S_\nu^*)S_\gamma = 0.$$

Proof: Exercise 2 c) on Sheet 13. ■

Lemma 18.8: *Let $k \in \mathbb{N}$. Then there is an isometry $w \in \mathcal{O}_n$ such that*

(i) *$wx = xw$ for all $x \in \mathcal{F}_k^n$,*

(ii) *For all multi-indices μ, ν with $|\mu|, |\nu| \leq k$, it holds*

$$w^*(S_\mu S_\nu^*)w = \begin{cases} S_\mu S_\nu^*, & \text{if } |\mu| = |\nu|, \\ 0 & \text{otherwise} \end{cases}$$

Proposition 18.9: *There is a faithful expectation*

$$\varphi \colon \mathcal{O}_n \longrightarrow \mathcal{F}^n,$$

where $\mathcal{F}^n = \mathrm{cl}(\mathrm{span}\{F_k^n \mid k \in \mathbb{N}\}) = \mathrm{span}\{S_\mu S_\nu^* \mid |\mu| = |\nu|\}$, with

$$\varphi(S_\mu S_\nu^*) = \begin{cases} S_\mu S_\nu^*, & \text{if } |\mu| = |\nu|, \\ 0, & \text{otherwise.} \end{cases}$$

and $\varphi(x) = w^* x w \in \mathcal{F}_k^n$ for $x \in \mathrm{span}\{S_\mu S_\nu^* \mid |\mu|, |\nu| \leq k\}$ and w from (Proposition 18.8).

Proof: For $\zeta \in \mathbb{S}^1$, the map

$$\rho_\zeta \colon \mathcal{O}_n \longrightarrow \mathcal{O}_n, \qquad S_i \longmapsto \zeta S_i$$

is an isomorphism with $\rho_{\overline{\zeta}} \circ \rho_\zeta = \mathrm{id}$. For $x \in \mathcal{O}_n$,

$$f_x \colon \mathbb{S}^1 \longrightarrow \mathcal{O}_n, \qquad \zeta \longmapsto \rho_\zeta(x)$$

is norm-continuous (like in the A_ϑ-proof). Then $\varphi(x) := \int_0^1 f_x(e^{2\pi \mathrm{i} t})\, dt$ is positive, linear, unital and faithful, where again

$$\frac{1}{n}\sum_{j=1}^{n} f_x(e^{2\pi \mathrm{i} t_j}) \longrightarrow \int_0^1 f_x(e^{2\pi \mathrm{i} t})\, dt\,.$$

It holds

$$\varphi(S_\mu S_\nu^*) = \int_0^1 \rho_{e^{2\pi \mathrm{i} t}}(S_\mu S_\nu^*)\, dt = \left(\int_0^1 e^{2\pi \mathrm{i} t(|\mu| - |\nu|)}\, dt\right) S_\mu S_\nu^* = \delta_{|\mu|,|\nu|} S_\mu S_\nu^*,$$

thus $\varphi^2 = \varphi$ and $\varphi(S_\mu S_\nu^*) = w^* S_\mu S_\nu^* w$ and $\varphi(S_\mu S_\nu^*) \in \mathcal{F}_k^n$ by (Lemma 18.6). ■

Definition 18.10: Let A be a unital C^*-algebra. A is purely infinite if and only if for all $0 \neq x \in A$ there are $a, b \in A$ such that $axb = 1$.

Remark 18.11: (i) If A is purely infinite, then A is simple. To see this, let $0 \neq I \triangleleft A$ be an ideal. Because A is purely infinite, for $0 \neq x \in I$, there are $a, b \in A$ such that $1 = axb \in I$, i. e. $I = A$.

(ii) The notion “purely infinite” comes from von Neumann algebras: If M is a von Neumann algebra of type III, then M has no finite projections. Let now A be a C^*-algebra. A is purely infinite if and only if it holds: “For all hereditary C^*-subalgebras $B \subseteq A$ (i. e., if $0 \leq a \leq b$ and $b \in B$, then $a \in B$), B has a finite projection”.

Theorem 18.12: *$\mathcal{O}_n$ is purely infinite (and thus in particular simple).*

Proof: (i) Let $0 \neq x \in \mathcal{O}_n$, then $x^*x \gneqq 0$, also $\varphi(x^*x) \gneqq 0$. Without loss of generality, $\|\varphi(x^*x)\| = 1$.

(ii) Find $y = y^* \in \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \text{ arbitrary}\}$ close to x^*, x, i. e., $\|x^*x - y\| < \frac{1}{4}$. This is possible: Because $\operatorname{cl}(\operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \text{ arbitrary}\}) = \mathcal{O}_n$, we find $y_0 \in \operatorname{span}\{S_\mu S_\nu^* \mid \mu, \nu \text{ arbitrary}\}$, such that $\|x^*x - y_0\| < \frac{1}{4}$; then put $y := 2^{-1}(y_0 + y_0^*)$.

(iii) Find $z \in \mathcal{O}_n$ such that $zyz^* = 1$.

Proof: We have $\|\varphi(y)\| > \frac{3}{4}$ since

$$1 = \|\varphi(x^*x)\| \leq \|\varphi(x^*x - y)\| + \|\varphi(y)\| < \frac{1}{4} + \|\varphi(y)\|.$$

Let now $y = \sum_{i=1}^n a_i S_{\mu_i} S_{\nu_i}^*$ and $k \geq \max\{|\mu_i|, |\nu_i|\}$. Thus there is an isometry $w \in \mathcal{O}_n$ with $\varphi(y) = w^*yw$. Since $\varphi(y) \in \mathcal{F}_k^n \cong M_{n^k}(\mathbb{C})$, we may view $\varphi(y)$ as a matrix which we can diagonalise, thus there exists a one-dimensional projection $e \in \mathcal{F}_k^n$ such that $e\varphi(y) = \varphi(y)e = \pm\|\varphi(y)\|e$ and a unitary $u \in \mathcal{F}_k^n$ such that $ueu^* = S_1^k S_1^{*k}$ (corresponding to $e_{1,1}$). Now put

$$z := \|\varphi(y)\|^{-\frac{1}{2}} S_1^{*k} uew^*.$$

It holds $z \in \mathcal{O}_n$ and $\|z\| < \frac{2}{\sqrt{3}}$, because

$$\|z\| \leq \|\varphi(y)\|^{-\frac{1}{2}} \|S_1^{*k} uew^*\| \leq \|\varphi(y)\|^{-\frac{1}{2}} < \frac{2}{\sqrt{3}},$$

and $zyz^* = 1$, because

$$zyz^* = \|\varphi(y)\|^{-1} S_1^{*k} uew^* y w e u^* S_1^k = \cdots = 1. \qquad \blacksquare$$

(iv) zx^*xz^* is invertible, since

$$\|1 - zx^*xz^*\| = \|z(y - x^*x)z^*\| \leq \|z\|^2\|y - x^*x\| \leq \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3} < 1.$$

Put $a := b^*x^*$, $b := z^*(zx^*xz^*)^{-\frac{1}{2}}$, then

$$axb = b^*x^*xb = (zx^*xz^*)^{-\frac{1}{2}} zx^*xz^* (zx^*xz^*)^{-\frac{1}{2}} = 1. \qquad \blacksquare$$

Remark 18.13: (i) There is a generalisation of $\mathcal{O}_n$ to graph C^*-algebras.

(ii) $\mathcal{O}_n \ncong \mathcal{O}_m$ for $n \neq m$.

19 Group C^*-algebras

Motivation: Let G be a locally compact group, furthermore let G be abelian. Then

$$\widehat{G} := \{\varphi \colon G \to \mathbb{T} \subseteq \mathbb{C} \text{ group homomorphism}\}$$

is the *dual group* of G. $\widehat{G}$ is again an abelian locally compact group. Pontrjagin duality states that

$$G \longrightarrow \hat{\hat{G}}, \qquad x \longmapsto \mathrm{ev}_x$$

is an isomorphism of topological groups. Hence G and $\widehat{G}$ are dual to each other. For example we have the following dual groups:

- $\widehat{\mathbb{Z}} = \mathbb{T}$ (think of the Fourier transformation),
- $\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z}$,
- $\widehat{\mathbb{R}} = \mathbb{R}$.

What about non-abelian groups G? Then $\widehat{G}$ has too little information and is no group in general. Instead, consider $\{G \to B(H) \text{ unitary representation}\}$ — note that if G is abelian and $\dim H = 1$, then $\{G \to B(H) \text{ unitary representation}\} = \widehat{G}$. By some Schur-Weyl / Tannaka-Krein / Peter-Weyl duality, we may reconstruct G from its representation theory.

Consider $C^*(G)$ or $C^*_{\mathrm{rad}}(G)$ whose representation theory is intuitively linked with the representation theory of G, i. e., group C^*-algebras arise from the idea to study the representation theory of groups by means from C^*-algebra theory. In fact, this was one of the reasons for introducing C^*-algebras as a concept (see J. Rosenberg, C^*-algebras and Mackey's theory of group representations, 1994 in Doran, R (ed.): C^*-algebras 1943-1993: A fifty year celebration, 1994).

By the way, one could also solve the problem of finding Pontrjagin duality in the non-abelian case by the following:

Sketch missing

which leads to topological quantum groups.

In conclusion: $C^*(G)$, $C^*_{\mathrm{rad}}(G)$ help to understand the representation theory of groups, but also they provide a huge class of examples of C^*-algebras.

Definition 19.1: Let G be a locally compact group. A *unitary representation of* G is a group homomorphism

$$\pi \colon G \longrightarrow \mathfrak{U}(H) := \{u \in B(H) \text{ unitary}\}$$

such that

$$G \longrightarrow H, \qquad g \longmapsto \pi(g)\xi$$

is continuous for all $\xi \in H$, i. e., π is continuous in the strong operator topology.

Remark 19.2: On $\mathfrak{U}(H)$, the strong— and the weak operator topology coincide. To see this, consider a net $(u_\lambda)_{\lambda\in\Lambda}$ in $\mathfrak{U}(H)$ such that $u_\lambda \to_{\mathrm{WOT}} u$. Then

$$\begin{aligned}\|u_\lambda\xi - u\xi\|^2 &= \|u_\lambda\xi\|^2 - \langle u_\lambda\xi, u\xi\rangle - \langle u\xi, u_\lambda\xi\rangle + \|u\xi\|^2 \\ &\to \|\xi\| - \|u\xi\|^2 - \|u\xi\|^2 + \|\xi\|^2 = 0.\end{aligned}$$

Theorem 19.3: *Let G be a locally compact group. Then there exists a left-invariant Radon measure (dt or μ_G) on G, unique up to a multiple by a constant. Hence for all $s \in G$ and $f \in L^1(G, dt)$ it holds*

$$\int_G f(t)\,dt = \int_G f(st)\,dt\,.$$

dt respectively μ_G is called the Haar measure*.*

Proof (idea): Construct on $C_c(G) := \{f\colon G \to \mathbb{C} \text{ continuous}, \operatorname{supp}(f) \text{ compact}\}$ a positive linear functional $\Lambda\colon C_c(G) \to \mathbb{C}$ such that $\Lambda(f) = \lambda({}_sf)$, where we denote ${}_sf(t) := f(st)$. By the Riesz representation theorem, there is a measure μ_G such that $\int_G f\,d\mu_G = \Lambda(f)$. ■

Remark 19.4: (i) If G is discrete, the Haar measure on G is the counting measure (up to normalisation $\mu_G(\{e\}) = 1$).

(ii) If G is compact, one usually normalises $\mu_G(G) = 1$.

(iii) In case $G = \mathbb{R}^n$, the Haar measure coincides with the Lebesgue measure.

Proposition 19.5: *The space $L^1(G) := L^1(G, dt)$ is a *-Banach algebra via*

$$(f \star g)(s) := \int_G f(t)g(t^{-1}s)\,dt, \quad f^*(s) := \Delta(s)^{-1}\overline{f(s^{-1})}, \quad \|f\|_1 := \int_G |f(t)|\,dt$$

for $f, g \in L^1(G)$ and $s \in G$. Here, $\Delta\colon G \to \mathbb{R}_+$ is the "modular function" with $\mu_G(E_s) = \Delta(s)\mu_G(E)$, where $E_s := \{ts \mid t \in E\}$ (note that μ_G is left-invariant, i. e., $\mu_G({}_sE) = \mu_G(E)$, but not right-invariant). G is called unimodular*, if $\Delta(s) \equiv 1$. $f \star g$ is called the* convulution *of f and g, which shall be the product on $L^1(G)$.*

Proof (Sketch): For $f, g \in L^1(G)$, Fubini's theorem yields

$$\begin{aligned}\|f \star g\|_1 &= \int_G |f \star g(s)|\,ds \\ &\le \int_G \int_G |f(t)||g(t^{-1}s)|\,ds\,dt = \int_G |f(t)\,dt \int_G |g(s)|\,ds = \|f\|_1\|g\|_1,\end{aligned}$$

thus $\|\cdot\|_1$ is submultiplicative with resect to the convolution and $f \star g \in L^1(G)$.

Δ is a group homomorphism, thus

$$f^{**}(s) = \Delta(s)^{-1}\overline{f^*(s^{-1})} = \Delta(s^{-1})\Delta(s^{-1})^{-1}f(s) = f(s).$$ ■

Remark 19.6: (i) If G is abelian, discrete or compact, then G is unimodular.

(ii) If G is discrete, then μ_G is the counting measure and for characteristic functions δ_t, $t \in G$, we have

$$(\delta_{t_1} \star \delta_{t_2})(s) = \sum_{g \in G} \delta_{t_1}(g)\delta_{t_2}(g^{-1}s) = \delta_{t_1,t_2}(s).$$

More generally,

$$\Big(\sum_i \alpha_i \delta_{t_i}\Big)\Big(\sum_j \beta_j \delta_j\Big) = \sum_{i,j} \alpha_i \beta_j \delta_{t_i,s_j},$$

thus convolution corresponds to multiplication of indices; furthermore $\delta_s^* = \delta_{s^{-1}}$.

(iii) We always have that $C_c(G) \subseteq L^1(G)$ is dense (the Haar measure is finite on compact sets).

(iv) Let G be discrete. Then

$$\mathbb{C}G = \Big\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C}, \alpha_g \neq 0 \text{ for finitely many } g \Big\}$$

is called the *group algebra*, that becomes an algebra with the multiplication

$$\Big(\sum_g \alpha_g g\Big)\Big(\sum_h \beta_h h\Big) = \sum_{g,h} \alpha_g \beta_h gh.$$

It holds $\mathbb{C}G = C_c(G)$, hence $L^1(G)$ is the completion of $\mathbb{C}G$ to a Banach algebra. There is an involution on $\mathbb{C}G$ via $\delta_g^* = \delta_{g^{-1}}$ (extend linearly).

Proposition 19.7: *Let G be locally compact.*

(i) *G is discrete if and only if $L^1(G)$ is unital; $L^1(G)$ is unital if and only if $\mu_G(\{e\}) \neq 0$ (δ_e then is the unit),*

(ii) *G is commutative if and only if $L^1(G)$ is commutative.*

Remark 19.8: $(L^1(G), \|\cdot\|_1)$ is no C^*-algebra (consider for instance the function $f = -\delta_{t^{-1}} + \delta_e + \delta_t$, then $\|f\|^2 = 9$, but $\|f^* \star f\| \leq 5$). Still, for any *-Banach algebra, there are two canonical way to turn it into a C^*-algebra (see Dixmier, 2.7.1).

Definition 19.9: Let G be a locally compact group. Put for $f \in L^1(G)$:

$$\|f\| := \sup\{\|\pi(f)\| \mid \pi \text{ representation of } L^1(G)\} \leq \|f\|_1.$$

The completion of $L^1(G)$ with respect to this norm is the *full (or maximal) group C^*-algebra* $C^*(G)$ or $C^*_{\max}(G)$ or $C^*_f(G)$.

Proposition 19.10: *Let G be a locally compact group.*

(i) *Every unitary representation $\pi\colon G \to B(H)$ induces a representation $\tilde{\pi}$ of $L^1(G)$.*

(ii) *The left regular representation*

$$\lambda\colon G \longrightarrow B(L^2(G)), \qquad (\lambda(s)f)(t) := f(s^{-1}t)$$

has an extension $\tilde{\lambda}$ to a faithful representation of $L^1(G)$.

(iii) *$\|\cdot\|$ from Definition 19.9 is indeed a norm.*

Proof (Sketch): (i) For $f \in L^1(G)$ put $\tilde{\pi}(f) := \int f(t)\pi(t)\,dt$, i. e.,

$$\langle \tilde{\pi}\xi, \eta\rangle := \int_G f(t)\langle \pi(t)\xi, \eta\rangle\,dt\,.$$

This $\tilde{\pi}$ is indeed a representation.

(ii) Use (i) and check for faithfulness.

(iii) Let $0 \neq x \in L^1(G)$. Then $\tilde{\lambda}(x) \neq 0$, but as $\tilde{\lambda}(x)$ appears in the supremum, $\|x\| \neq 0$. The rest is clear, because the supremum is taken over C^*-seminorms. ■

Definition 19.11: The reduced group C^*-algebra $C^*_{\text{rad}}(G)$ or $C^*_{\min}(G)$ or $C^*_r(G)$ is given by

$$C^*_{\text{red}}(G) := \text{cl}(\tilde{\lambda}(L^1(G))) \subseteq B(L^2(G)).$$

We always have a homomorphism $C^*(G) \to C^*_{\text{red}}(G)$ which is an isomorphism if and only if G is amenable and $L(G) = C^*_{\text{red}}(G)'' \subseteq B(L^2(G))$.

Remark 19.12: Let G be discrete.

(i) Then G is locally compact and the Haar measure is the counting measure. Also, G is unimodular.

(ii) We can define $C^*_{\max}(G) := \text{cl}_{\|\cdot\|}(\mathbb{C}G)$ with the norm

$$\|x\| := \sup\{\|\pi(x)\| \mid \pi \text{ is } {}^*\text{-representation of } \mathbb{C}G\} < \infty,$$

and $C^*_{\max} = \text{cl}_{\|\cdot\|}(L^1(G))$ from Definition 19.9 coincides with this definition, because $\mathbb{C}G = C_c(G) \subseteq L^1(G)$ is dense.

(iii) We have $C^*_{\max}(G) \cong C^*(u_g, g \in G \mid u_g \text{ unitary}, u_g u_h = u_{gh}, u_g^* = u_{g^{-1}})$, where $u_e = 1$ in the universal C^*-algebra.

Proof: By the universal property, we have a map

$$\varphi\colon C^*(u_g, g \in G \mid u_g \text{ unitary}, u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}) \longrightarrow C^*_{\max}(G), \qquad u_g \longmapsto \delta_g.$$

Note that $u_e u_g = u_g u_e = u_g$, $u_e^2 = u_e$, thus u_e is the unit. φ is surjective, since $\mathbb{C}G \subseteq \operatorname{im}(\varphi)$ and $\mathbb{C}G$ is dense. For the injectivity of φ, let

$$\pi\colon C*(u_g, g \in G \mid u_g \text{ unitary}, u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}) \longrightarrow B(H)$$

be a faithful representation. Then

$$\alpha^\circ\colon G \longrightarrow B(H), \qquad g \longmapsto \pi(u_g)$$

is a unitary representation of G, thus there is an extension $\alpha\colon C^*_{\max}(G) \to B(H)$, hence

$$\begin{tikzcd}
C^*(u_g, g \in G \mid \dots) \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & B(H) \\
& C^*_{\max}(G) \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

commutes and if $\varphi(x) = 0$, then $\pi(x) = \alpha \circ \varphi(x) = 0$ and thus, as π is injective, $x = 0$. ■

(iv) If $\varphi\colon G \to B(H)$ is a unitary representation of the group G, then

$$\tilde{\varphi}\colon C^*_{\max}(G) \longrightarrow B(H), \qquad \delta_g \longmapsto \varphi(g),$$

is a representation the full group-C^* algebra. Conversely, if a representation $\tilde{\varphi}\colon C^*_{\max}(G) \to B(H)$ is given, then we get a representation of the group G via

$$\varphi\colon G \longrightarrow B(H), \qquad g \longmapsto \tilde{\varphi}(\delta_g).$$

(v) Let $\lambda\colon G \to B(\ell^2(G))$ be a left regular representation. Then $\lambda(g)\delta_h = \delta_{gh}$ is just left multiplication. Thus there is a faithful representation of $\mathbb{C}G$ via

$$\tilde{\lambda}\colon \mathbb{C}G \longrightarrow B(\ell^2(G)), \qquad \sum \alpha_g \delta_g \longmapsto \sum \alpha_g \lambda(g).$$

To see that $\tilde{\lambda}$ is faithful, let $x = \sum \alpha_g \delta_g \neq 0$, i. e., there is g_0 such that $\alpha_{g_0} \neq 0$. Then

$$\langle \tilde{\lambda}(x)\delta_e, \delta_{g_0} \rangle = \left\langle \sum \alpha_g \delta_g, \delta_{g_0} \right\rangle = \alpha_{g_0} \neq 0.$$

Because $\tilde{\lambda}$ is faithful, we have $\mathbb{C}G \cong \tilde{\lambda}(\mathbb{C}G) \subseteq B(\ell^2(G))$ and thus can define $C^*_{\text{red}}(G) := \operatorname{cl}(\tilde{\lambda}(\mathbb{C}G)) \subseteq B(\ell^2(G))$.

(vi) By (iv), the left regular representation extends to $\tilde{\lambda}\colon C^*_{\max}(G) \to B(\ell^2(G))$, so in fact $\tilde{\lambda}\colon C^*_{\max}(G) \to C^*_{\text{red}}$ is a surjective *-homomorphism, $\tilde{\lambda}$ is no isomorphism. One can show there are surjective *-homomorphisms

$$C^*_{\max}(G) \longrightarrow \operatorname{cl}_{\|\cdot\|}(\mathbb{C}G) \longrightarrow C^*_{\text{red}},$$

identity on $\mathbb{C}G$, hence: $C^*_{\max}(G)$ is the maximal— and $C^*_{\min}$ is the minimal C^*-completion of $\mathbb{C}G$. This map from $C^*_{\max}(G)$ to C^*_{red} exists for general locally compact groups and we have that $\tilde{\lambda}$ is an isomorphism if and only if G is amenable. G is called *amenable*, if there is a state $m\colon L^\infty(G) \to \mathbb{C}$ such that $m(f_s) = m(f)$ for all $f \in L^\infty(G)$, $s \in G$ and $f_s(t) := f(s^{-1}t)$.[1] For example: If G is compact, finite or abelian, then G is amenable.

(vii) $L(G) := \tilde{\lambda}(\mathbb{C}G)'' = C^*_{\text{red}}(G)'' \subseteq B(\ell^2(G))$ is the group von Neumann algebra for a group G.

(viii) If G is abelian and locally compact, then $C^*_{\max}(G) \cong C_0(\widehat{G})$.

(ix) There also are right regular representations, but those yield the same C^*-algebras.

(x) We have a trace $\tau(x) = \langle x\delta_e, \delta_e\rangle$ on $C^*_{\text{red}}(G)$ by (Theorem 11.8).

Example 19.13: Consider $G = \mathbb{Z}$. Then

$$C^*(\mathbb{Z}) = C^*(u_n, n \in \mathbb{Z} \mid u_n \text{ unitary}, u_n u_m = u_{n+m}, u_{-m} = u_m^*),$$

hence $u_0 = 1$, $u_n = (u_1)^n$ for $n > 0$ and $u_n = (u_1^*)^{|n|}$ for $n < 0$. Thus we have the equalities $C^*(\mathbb{Z}) = C^*(u_1 \text{ unitary}) = C(\mathbb{S}^1) = C(\widehat{\mathbb{Z}})$. Because $\mathbb{Z}$ is abelian, $\mathbb{Z}$ is amenable, hence $C^*_{\max}(\mathbb{Z}) = C^*_{\text{red}}(\mathbb{Z})$.

Example 19.14: Consider $G = \mathbf{F}_2$, the free group on two generators x and y. This G is not amenable, thus $C^*_{\max}(\mathbf{F}_2) \neq C^*_{\text{red}}(\mathbf{F}_2)$. Also, we can write $C^*_{\max}(\mathbf{F}_2)$ as universal C^* algebra: $C^*_{\max}(\mathbf{F}_2) = C^*(u, v \text{ unitaries})$.

These group C^*-algebras and the von Neumann algebras $L(\mathbf{F}_n)$ on n generators are famous. Some properties of these are:

- $C^*(\mathbf{F}_n)$ and $C^*_{\text{red}}(\mathbf{F}_n)$ have faithful traces,
- $C^*(\mathbf{F}_n)$ and $C^*_{\text{red}}(\mathbf{F}_n)$ have no projections (a hot topic was to find such examples until 1981[2]).
- $C^*_{\text{red}}(\mathbf{F}_n)$ is simple.

The problem for von Nemann algebras is the following: It is well known that $\mathbf{F}_n \ncong \mathbf{F}_m$, if $n \neq m$. Furthermore it can be shown that for $n \neq m$, also $\mathbb{C}\mathbf{F}_n \ncong \mathbb{C}\mathbf{F}_m$ and $C^*_{\text{red}}(\mathbf{F}_n) \ncong C^*_{\text{red}}(\mathbf{F}_m)$. But for the group von Neumann algebras, it is still unknown whether $L(\mathbf{F}_n) \ncong L(\mathbf{F}_m)$ or $L(\mathbf{F}_n) \cong L(\mathbf{F}_m)$. This problem is called the free group factor problem and it has been open for more than 80 years now.

[1] This is only one of very many possible definitions for "amenable".

[2] Kaplansky started the interest in this question in 1958, when he first asked for an example of a simple C^*-algebra without projections.

20 Products of C^*-algebras

Given two C^*-algebras A, B. How can we form a "product" $A \; ? \; B$?

Definition 20.1: Let A, B be unital C^*-algebras and C be a C^*-algebra, $j_1 \colon C \hookrightarrow A$, $j_2 \colon C \hookrightarrow B$ two embeddings.

(i) The C^*-algebra

$$A *_{\mathbb{C}} B := C^*(a \in B, b \in B \mid \text{relations of } A, \text{ relations of } B, 1_A = 1_B)$$

is called the *(unital) free product of A and B*,

(ii) The C^*-algebra

$$A *_C B := C^*(a \in A, b \in B \mid \text{relations of } A, \text{ relations of } B, j_1(x) = j_2(x) \text{ for all } x \in C\}$$

is called the *amalgamated free product of A and B*.

Proposition 20.2: *The free product has the following universal property:*

$$\begin{tikzcd}
 & A \arrow[dr] & \\
C \arrow[ur, "j_1"] \arrow[r] \arrow[dr, "j_2"'] & A *_C B \arrow[r] & D \\
 & B \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Example 20.3: We have

$$C^*_{\max}(\mathbf{F}_2) = C(\mathbb{S}^1) * C(\mathbb{S}^1) = C^*(\mathbb{Z}) * C^*(\mathbb{Z}) = C^*(\mathbb{Z} * \mathbb{Z})$$

Remark 20.4: There are reduced free products.

Remark 20.5: Let A, B be C^*-algebras. Then

$$A \odot B := \operatorname{span}\Big\{ \sum_{i=1}^n a_i \otimes b_i : a_i \in A, b_i \in B, n \in \mathbb{N} \Big\} \Big/ \begin{aligned} \big((x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y, \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2, \\ \lambda(x \otimes y) &= (\lambda x) \otimes y = x \otimes (\lambda y)\big) \end{aligned}$$

is the *algebraic tensor product* of A and B.

Our concern shall now be to find a C^*-norm on $A \odot B$, of which there are in fact many.

Definition 20.6: Let A, B be C^*-algebras. For $\sum_{i=1}^n a_i \otimes b_i \in A \odot B$ put

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{\max} := \sup\{\|\pi(x)\| \mid \pi\colon A \odot B \longrightarrow B(H)^*\text{-homomorphism}\},$$

then $A \otimes_{\max} B := \mathrm{cl}_{\|\cdot\|_{\max}}(A \odot B)$ is called the *maximal tensor product of A and B;*

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_{\min} := \Big\| \sum_{i=1}^n \pi(a_i) \otimes \sigma(b_i) \Big\|,$$

where $\pi\colon A \hookrightarrow B(H)$, $\sigma\colon B \hookrightarrow B(K)$ are faithful representations and $\pi \otimes \sigma\colon A \odot B \to B(H \otimes K)$. Then $A \otimes_{\min} B := \mathrm{cl}_{\|\cdot\|_{\min}}(A \odot B)$ is called the *minimal- or spatial tensor product.*

Proposition 20.7: (i) *We have a universal property:*

$$\begin{tikzcd} A \odot B \arrow[r] \arrow[d] & K \\ A \otimes_{\max} B \arrow[ur] & \end{tikzcd}$$

(ii) *For all C^*-norms $\|\cdot\|_\alpha$ on $A \odot B$ we have the diagram*

$$\begin{tikzcd} A \otimes_{\max} B \arrow[r] & A \otimes_{\|\cdot\|_\alpha} B \arrow[r] & A \otimes_{\min} B \\ A \odot B \arrow[r, "\mathrm{id}"'] \arrow[u] & A \odot B \arrow[r, "\mathrm{id}"'] \arrow[u] & A \odot B \arrow[u] \end{tikzcd}$$

Proposition 20.8: *Let A, B be unital C^*-algebras. Then*

$$A \otimes_{\max} B = C^*(a \in A,\ \textit{relations of } A, b \in B \textit{ relations of } B \mid ab = ba, 1_A = 1_B).$$

Example 20.9: We have

$$C^*(\mathbb{Z}^2) = C^*(\mathbb{Z} \times \mathbb{Z}) = C^*(\mathbb{Z}) \otimes C^*(\mathbb{Z}) = C(\mathbb{S}^1) \otimes C(\mathbb{S}^1).$$

Remark 20.10: A C^*-algebra A is called *nuclear*, if for all C^*-algebras B it holds $A \otimes_{\min} B = A \otimes_{\max} B$.

If A is commutative, then A is commutative (for example, $\mathcal{T}$, A_ϑ and $\mathcal{O}_n$ are nuclear).

G is amenable if and only if $C^*(G)$ is nuclear.

Remark 20.11: Recall: For A, B C^*-algebras,

$$A \oplus B := \{(a,b) \mid a \in A, b \in B\}, \qquad \|(a,b)\| := \max\{\|a\|, \|b\|\}$$

is called the direct sum of the C^*-algebras A and B. We can express $A \oplus B$ as universal C^*-algebra via

$$A \oplus B = C^*(a \in A, \text{ relations of } A, b \in B, \text{relations of } B \mid ab = 0).$$

Definition 20.12: Let G be a locally compact group, A a C^*-algebra and let $\alpha\colon G \to \operatorname{Aut}(A), g \mapsto \alpha_g$ be a continuous group homomorphism (i. e., $g \mapsto \alpha_g(x)$ is continuous for all $x \in A$). Then α is called an *action of G on A*, A is called a G-C^*-algebra, the triple (A, G, α) is called a *covariant system or C^*-dynamical system.*

Definition 20.13: A covariant representation of a C^*-dynamical system (A, G, α) is a non-degenerate representation $\pi\colon A \to B(H)$ together with a unitary representation $G \to \mathfrak{U}(H)$, $g \mapsto u_g$ (on the same Hilbert space!) such that for all $a \in A$, $g \in G$, it holds

$$\pi(\alpha_g(x)) = u_g \pi(x) u_g^{-1}.$$

Remark 20.14: In the above situation, the unitaries $(u_g)_{g\in G}$ implement the automorphisms $(\alpha_g)_{g \in G}$, i. e., they make them inner.

Indeed, consider a C^*-algebra $A \subseteq B(H)$ and a unitary $u \in A$. Then

$$A \longrightarrow A, \qquad u \longmapsto uxu^*$$

is a *-homomorphism ($x \mapsto u^*xu$ is the inverse of $x \mapsto uxu^*$). It is an inner automorphism of A.

Remark 20.15: Let G be discrete. Consider

$$AG := \Big\{ \sum_{g \in G}^{\text{fin}} a_g \delta_g : a_g \in A \Big\}$$

similar to the group algebra but with coefficients from A rather than from $\mathbb{C}$, with twisted multiplication

$$(\delta_{t_1} * a\delta_{t_2}) := \alpha_{t_1}(a)\delta_{t_1 t_2}, \qquad \delta_t^* := \delta_{t^{-1}}$$

or rather

$$\Big(\sum a_t \delta_t\Big)\Big(\sum b_s \delta_s\Big) := \sum_{s,t} a_t \alpha_t(b_s) \delta_{ts}.$$

How does this make $(\alpha_t)_{t \in G}$ inner? We have

$$\Big(\sum_t a_t \delta_t\Big)\Big(\sum_s b_s \delta_s\Big) = \sum_{s,t} a_t \delta_t b_s \delta_t^* \delta_t \delta_s = \sum_{s,t} a_t \alpha_t(b_s) \delta_s.$$

This construction works analogously for locally compact groups using $L^1(G)$ instead of $\mathbb{C}G$.

Definition 20.16: Let G be discrete. Put

$$\begin{aligned}\|x\| := \sup\{\|\pi(x)\| \mid \pi\colon AG \to B(H) \text{ non-degenerate} \\ {}^*\text{-homomorphism induced from a covariant representation of } (A,G,\alpha)\}\end{aligned}$$

and call $A \rtimes_\alpha G := \mathrm{cl}_{\|\cdot\|}(AG)$ the *cross-product of A with G (given α).*

Remark 20.17: One can define left regular representations for C^*-dynamical systems and one does obtain a reduced version $A \rtimes_r G$.

Proposition 20.18: *Let G be discrete, A be a unital C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, i. e., let (A,G,α) be a C^*-dynamical system. Then*

$$\begin{aligned}A \rtimes_\alpha G = C^*(&a \in A, \textit{relations of } A, u_g \textit{ unitaries}, g \in G \mid \\ &u_g u_h = u_{gh}, u_g^* = u_{g^{-1}}, \alpha_g(a) = u_g a u_g^* \textit{for all } g,h \in G,\ a \in A, 1_A = u_e)\end{aligned}$$

Remark 20.19: (i) $A \rtimes_\alpha G$ contains a copy of A, a copy of G and it makes the automorphism α_g inner. It is the smallest C^*-algebra with this property. One can also say that A is obtained by adjoining unitaries making the α_g inner.

Example 20.20: (i) Let X be a locally compact space, G be a locally compact group acting on X, i. e.,

$$\alpha^\circ\colon X \times G \longrightarrow X, \qquad (x,g) \longmapsto gx.$$

Then $(C_0(X), G, \alpha)$ is a C^*-dynamical system with

$$\alpha\colon G \longrightarrow \mathrm{Aut}(C_0(X)), \qquad \alpha_g(f)(x) := f(\alpha^\circ(x,g)) = f(gx)$$

for $x \in X, g \in G$. This classical understanding of a dynamical system matches the notion of a C^*-dynamical system.

(ii) If $A = \mathbb{C}$, then $\alpha\colon G \to \mathrm{Aut}(G)$ acts trivially via $\alpha_g(\lambda) = \lambda$. Then for the cross-product $A \rtimes_\alpha G$ it holds: $A \rtimes_\alpha G = \mathbb{C} \rtimes_\alpha G = C^*(G)$.

More generally: If G acts trivially of some C^*-algebra A (i. e., $\alpha_g \equiv \mathrm{id}$), then $A \rtimes_\alpha G \cong A \otimes_{\max} C^*(G)$. We then also have $A \rtimes_\alpha^r G \cong A \otimes_{\min} C^*_{\mathrm{red}}(G)$.

(iii) We have $C^*(N \rtimes_\alpha H) = C^*(N) \rtimes_\alpha H$, where on the left side we mean the semidirect product of groups and on the right side the cross-product of C^*-algebras.

(iv) For $G = \mathbb{Z}$, the action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ consists in one single automorphism α_1, since $\alpha_n = \alpha_1 \circ \cdots \circ \alpha_1$ and every automorphism $\alpha\colon A \to A$ induces an action

$$\mathbb{Z} \longrightarrow \mathrm{Aut}(A), \qquad u \longmapsto \alpha \circ \cdots \circ \alpha.$$

Then

$$A \rtimes_\alpha \mathbb{Z} = C^*(a \in A, \text{relations of } A, u \text{ unitary} \mid uau^* = \alpha(a)).$$

If for instance $A = C(\mathbb{S}^1)$ and

$$\alpha\colon C(\mathbb{S}^1) \longrightarrow C(\mathbb{S}^1) = C^*(u \text{ unitary}), \qquad u \longmapsto e^{2\pi i \vartheta} u,$$

then $C(\mathbb{S}^1) \rtimes_{\alpha_\vartheta} \mathbb{Z} = A_\vartheta$ is the rotation algebra.