**UNIVERSITÄT DES SAARLANDES**
**FACHRICHTUNG MATHEMATIK**
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**Exercises for the lecture *Operator algebras (Functional analysis II)***
Summer term 2018

**Sheet 1**
**submission:** Monday, April 16 2018, 2 pm
postbox of Andreas Widenka (basement of building E2.5)

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**Exercise 1** (20 + 5* points)**.** On $\ell^2 = \ell^2(\mathbb{N})$, fix the standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and consider the associated *unilateral shift* $S \in B(\ell^2)$, which is determined by $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$.

(a) We define $f_{ij} := S^{i-1}(1-SS^*)(S^*)^{j-1}$ for all $i, j \in \mathbb{N}$. Prove the following statements:

  (i) $f_{ii}$ is a projection of rank 1,

  (ii) $f_{ij}f_{kl} = \delta_{jk}f_{il}$,

  (iii) $f_{ij}^* = f_{ji}$,

  (iv) $f_{ij}e_n = \delta_{jn}e_i$.

  In the following, we put $M_n := C^*(f_{ij} \mid 1 \leq i, j \leq n) \subseteq B(\ell^2)$.

(b) Let $E_{ij}$ denote the matrix where the $ij$-th entry is 1 and all other entries are 0. Show that the map

$$\psi : \; M_n(\mathbb{C}) \to M_n, \quad \sum_{i,j=1}^{n} \alpha_{ij}E_{ij} \mapsto \sum_{i,j=1}^{n} \alpha_{ij}f_{ij}$$

  is a $*$-isomorphism.

  **Hint**: Functional analysis I, Sheet 10, Exercise 3

(c) Let $T \in B(\ell^2)$ be an operator with the property that the image of both $T$ and $T^*$ is contained in $\operatorname{span}\{e_1, \ldots, e_n\} \subset \ell^2$ for some $n \in \mathbb{N}$. Show that $T$ can be written as $T = \sum_{i,j=1}^{n} \alpha_{ij}f_{ij}$ for some $\alpha_{ij} \in \mathbb{C}$.
  Show that an arbitrary finite rank operator $T \in B(\ell^2)$ can be approximated in operator norm by a sequence $(T_n)_{n=1}^{\infty}$ of finite rank operators of the previous form.

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(d) See Remark 9.9 in the lecture notes of *Functional analysis I* for the definition of the strong operator topology (SOT). We define

$$\begin{aligned} *\text{-alg}(S) &:= \{\text{n.c. polynomials in } S \text{ and } S^*\}, \\ C^*(S) &:= \overline{*\text{-alg}(S)}^{\|\cdot\|}, \\ W^*(S) &:= \overline{*\text{-alg}(S)}^{\text{SOT}}. \end{aligned}$$

Show that $\mathcal{K}(\ell^2) \subseteq C^*(S) \subseteq W^*(S) = B(\ell^2)$.

**Hint**: You may use Theorem 9.8 and Remark 9.9 in the lecture notes of *Functional analysis I* without proving them.

(e)* Are the inclusions $\mathcal{K}(\ell^2) \subseteq C^*(S)$ and $C^*(S) \subseteq W^*(S)$ proper?

**Exercise 2** (20 points)**.** Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and denote by $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$. The *weak topology on $H$* is the locally convex topology on $H$ that is generated by the family $\mathcal{P} = \{p_x \mid x \in H\}$ of seminorms

$$p_x : \; H \to [0, \infty), \quad y \mapsto |\langle y, x \rangle|.$$

For details on that terminology, we refer to Definitions 1.17 and 1.33 in the lecture notes of *Functional analysis I.*

(a) Verify that the weak topology on $H$ is Hausdorff.

(b) Prove that every bounded sequence $(x_n)_{n=1}^\infty$ in $H$ has a weakly convergent subsequence $(x_{n_k})_{k=1}^\infty$.

**Hint:** Assume first that $H$ is separable. Construct a countable subset $\mathcal{F}$ of $H'$ which is dense in $H'$ with respect to $\|\cdot\|_{H'}$. Use a diagonal argument to extract a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ for which $(f(x_{n_k}))_{k=1}^\infty$ is convergent for all $f \in \mathcal{F}$. Deduce that $(f(x_{n_k}))_{k=1}^\infty$ is in fact convergent for all $f \in H'$. Use the reflexivity of $H$ to find the weak limit of $(x_{n_k})_{k=1}^\infty$. Finally, reduce the general case to the previously discussed case of a separable Hilbert space.

(c) Prove that every bounded sequence $(x_n)_{n=1}^\infty$ in $H$ has a weakly convergent subsequence $(x_{n_k})_{k=1}^\infty$ for which the sequence

$$\left( \frac{1}{K} \sum_{k=1}^{K} x_{n_k} \right)_{K=1}^{\infty}$$

is convergent in the norm topology.

**Hint:** Use (b) in order to reduce the problem to the case where $(x_n)_{n=1}^\infty$ is itself weakly convergent and has the weak limit 0. Construct iteratively a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that

$$|\langle x_{n_{k+1}}, x_{n_l} \rangle| \leq \frac{1}{k} \qquad \text{for all } k \in \mathbb{N} \text{ and } l = 1, \ldots, k.$$

Finally, expand $\left\| \frac{1}{K} \sum_{k=1}^{K} x_{n_k} \right\|^2$ and use the properties of $(x_{n_k})_{k=1}^\infty$ in order to show that $\frac{1}{K} \sum_{k=1}^{K} x_{n_k}$ converges to 0 in norm as $K \to \infty$.