Exercise 1 (20 points). Let $A$ be a non-unital $C^*$-algebra. A double centralizer of $A$ is a pair $(L, R)$ of linear maps from $A$ to $A$ satisfying $L(ab) = L(a)b, R(ab) = aR(b)$ and $aL(b) = R(a)b$ for all $a, b \in A$. The multiplier algebra $M(A)$ of $A$ is defined as follows:

$$M(A) := \{(L, R) \text{ double centralizer of } A\}$$

(a) Show that $M(A)$ is a unital $C^*$-algebra and that the map

$$A \to M(A), \quad a \mapsto (L_a, R_a)$$

is an isometric $*$-homomorphism. Furthermore prove that $A$ is an ideal in $M(A)$. Thus every non-unital $C^*$-algebra can be embedded as an ideal into a unital $C^*$-algebra.

(b) Prove that $M(A)$ is the largest unitization of $A$: If $B$ is a unital $C^*$-algebra and $A \subseteq B$ as an ideal, then there is a $*$-homomorphism from $B$ to $M(A)$ that extends the embedding $A \subseteq M(A)$.

Exercise 2 (20 points). Let $X$ be a locally compact metric space and let $A = C_0(X)$.

(a) Show that $A$ is a commutative $C^*$-algebra.

(b) Prove that $C_0(\mathbb{R})$ is not unital.

(c) The one-point compactification $\hat{X}$ of $X$ is given by the set $\hat{X} := X \cup \{\infty\}$ together with the following topology: A set $U \subseteq \hat{X}$ is a neighbourhood of $x \in X$ if and only if $U \cap X$ is a neighbourhood of $x$. A set $U \subseteq \hat{X}$ is a neighbourhood of $\infty$ if and only if $U$ contains the complement of a compact set in $X$.

Prove $\hat{A} = C(\hat{X}) = A \oplus \mathbb{C}1$.

(d) We define $C_b(X) := \{f : X \to \mathbb{C} \text{ continuous, bounded}\}$. Show $M(A) = C_b(X)$.

(e) Verify that $C(\hat{X})$ is a proper subset of $C_b(X)$.