UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Prof. Dr. Moritz Weber
Dr. Tobias Mai
M.Sc. Simon Schmidt

Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 2
submission: Monday, April 23 2018, before the lecture

Exercise 1 (20 points)**.** Let A be a non-unital C^*-algebra. A *double centralizer* of A is a pair (L, R) of linear maps from A to A satisfying $L(ab) = L(a)b, R(ab) = aR(b)$ and $aL(b) = R(a)b$ for all $a, b \in A$. The *multiplier algebra* $M(A)$ of A is defined as follows:

$$M(A) := \{(L, R) \text{ double centralizer of } A\}$$

(a) Show that $M(A)$ is a unital C^*-algebra and that the map

$$A \to M(A), \qquad a \mapsto (L_a, R_a)$$

is an isometric $*$-homomorphism. Furthermore prove that A is an ideal in $M(A)$. Thus every non-unital C^*-algebra can be embedded as an ideal into a unital C^*-algebra.

(b) Prove that $M(A)$ is the largest unitization of A: If B is a unital C^*-algebra and $A \subseteq B$ as an ideal, then there is a $*$-homomorphism from B to $M(A)$ that extends the embedding $A \subseteq M(A)$.

Exercise 2 (20 points)**.** Let X be a locally compact metric space and let $A = C_0(X)$.

(a) Show that A is a commutative C^*-algebra.

(b) Prove that $C_0(\mathbb{R})$ is not unital.

(c) The *one-point compactification* $\hat{X}$ of X is given by the set $\hat{X} := X \cup \{\infty\}$ together with the following topology: A set $U \subseteq \hat{X}$ is a neighbourhood of $x \in X$ if and only if $U \cap X$ is a neighbourhood of x. A set $U \subseteq \hat{X}$ is a neighbourhood of ∞ if and only if U contains the complement of a compact set in X.

Prove $\tilde{A} = C(\hat{X}) = A \oplus \mathbb{C}1$.

(d) We define $C_b(X) := \{f : X \to \mathbb{C} \text{ continuous, bounded}\}$. Show $M(A) = C_b(X)$.

(e) Verify that $C(\hat{X})$ is a proper subset of $C_b(X)$.