UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Tobias Mai
Prof. Dr. Moritz Weber
M.Sc. Simon Schmidt

Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 3
submission: Monday, April 30 2018, before the lecture

Exercise 1 (10 points). Let A be a C^*-algebra, $a \in A$ normal and $f \in C(\mathrm{Sp}(a))$ a continuous function on the spectrum of a. Prove that $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$, where $f(\mathrm{Sp}(a)) = \{f(\lambda) \mid \lambda \in \mathrm{Sp}(a)\}$.

Exercise 2 (10 points). Let A be a unital Banach algebra and $x, y \in A$. Prove that

$$\mathrm{Sp}(xy) \cup \{0\} = \mathrm{Sp}(yx) \cup \{0\}.$$

Exercise 3 (10 points).

(a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be functions in $C_0(\mathbb{R})$, where $f_n(x) = 1$ for $|x| < n$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is an approximate unit for $C_0(\mathbb{R})$.

(b) Let H be a separable Hilbert space of infinite dimension. Find a sequence of operators $(p_n)_{n\in\mathbb{N}}$ which is an approximate unit for the compact operators $\mathcal{K}(H)$.

Exercise 4 (10 points). Let A be a C^*-algebra and $x, h \in A$ selfadjoint with $h \geq 0$, $h \geq x$. The positive part x_+ of x is defined as $x_+ := f_+(x)$ via the functional calculus, where we have $f_+ : \mathbb{R} \to \mathbb{R}$, $f_+(t) := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

(a) Show that $h \geq x_+$ if A is commutative.

(b) Verify that $h \ngeq x_+$ in general by giving a counterexample in the case $A = M_2(\mathbb{C})$.