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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 4
submission: Monday, May 7 2018, before the lecture

Exercise 1 (10 points)**.** Let tr be the normalized trace on $M_n(\mathbb{C})$, given by $\mathrm{tr}((a_{ij})) = \frac{1}{n}\sum_{i=1}^n a_{ii}$, where we know $\mathrm{tr}(xy) = \mathrm{tr}(yx)$. Let $h \in M_n(\mathbb{C})$ be positive and $f_h : M_n(\mathbb{C}) \to \mathbb{C}$ be the map defined by $f_h(x) := \mathrm{tr}(hx)$. Show that the map $h \mapsto f_h$ is an orderpreserving bijection between the positive elements of $M_n(\mathbb{C})$ and the positive functionals on $M_n(\mathbb{C})$, with $\|f_h\| = \mathrm{tr}(h)$.

Exercise 2 (10 points)**.** Let f be a state on a C^*-algebra A and (π_f, H_f) its GNS representation.

(a) Let $I \lhd A$ be an ideal in A. Prove that $I \subseteq \ker(\pi_f)$ if and only if $I \subseteq \ker(f)$.

(b) The state f is called *faithful*, if $f(a) = 0$ implies $a = 0$ for all positive elements $a \in A$. Show that $\ker(\pi_f) = 0$ if f is faithful.

(c) Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. Verify that $(\pi_f(u_\lambda))_{\lambda\in\Lambda}$ converges to 1 in the strong operator topology on $B(H_f)$, i.e. $\pi_f(u_\lambda)\xi \to \xi$ for all $\xi \in H_f$.

Exercise 3 (10 points)**.** Let (π_i, H_i, ξ_i), $i = 1, 2$ be cyclic representations of a C^*-algebra A and $f_i : A \to \mathbb{C}$, $i = 1, 2$ positive functionals with $f_i(x) = \langle \pi_i(x)\xi_i, \xi_i\rangle$. Prove that if $f_1 = f_2$, then there exists a unitary $U : H_1 \to H_2$ with $\pi_2(x) = U\pi_1(x)U^*$ and $U\xi_1 = \xi_2$.

Exercise 4 (10 points)**.** Let $e, f \in B(H)$ be projections. Show that the following are equivalent:

(i) $ef = e$

(ii) $fe = e$

(iii) $eH \subseteq fH$

(iv) $f - e$ is a projection

(v) $f - e$ is positive

If one (and hence all) of these conditions is satisfied, we write $e \leq f$.