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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 5
submission: Monday, May 14 2018, before the lecture

Exercise 1 (10 points). Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space.

(a) Work out the details of Remark 1.2, i.e., show that $\tau_{\text{WOT}} \subseteq \tau_{\text{SOT}} \subseteq \tau_{\text{ONT}}$.

(b) Prove that $x \mapsto x^*$ is continuous with respect to ONT and WOT.

(c) Prove Remark 1.3 (iii), i.e., show that for fixed $y \in B(H)$ the linear mappings $x \mapsto xy$ and $x \mapsto yx$ are both continuous with respect to SOT and WOT.

(d) Show that if A is a $*$-subalgebra of $B(H)$, then so is $\overline{A}^{\text{WOT}}$.

Exercise 2 (10 points). Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space.

(a) Consider a strongly continuous linear functional $\varphi : B(H) \to \mathbb{C}$. Show that there are $n \in \mathbb{N}$, vectors $\xi_1, \dots, \xi_n \in H$, and some $C > 0$ such that

$$|\varphi(x)| \leq C \left(\sum_{i=1}^{n} \|x\xi_i\|^2 \right)^{1/2} \qquad \text{for all } x \in B(H).$$

(b) Prove Lemma 6.2 of the lecture: Let $\varphi : B(H) \to \mathbb{C}$ be a linear functional. Then the following statements are equivalent:

(i) φ is continuous with respect to the weak operator topology.

(ii) φ is continuous with respect to the strong operator topology.

(iii) There are $n \in \mathbb{N}$ and vectors $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in H$, such that

$$\varphi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i \rangle \qquad \text{for all } x \in B(H).$$

Hint: Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). For proving (ii) $\Rightarrow$ (iii), use (a) and consider the Hilbert subspace $K \subseteq H^n$ that is given as the closure of $K_0 := \{(x\xi_1, \dots, x\xi_n) \mid x \in B(H)\}$; show that the assignment $(x\xi_1, \dots, x\xi_n) \mapsto \varphi(x)$ is well-defined on K_0 and extends to a continuous linear functional $\psi : K \to \mathbb{C}$; finally apply the Riesz representation theorem to ψ.

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Exercise 3 (10 points)**.** Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and let $S \subseteq B(H)$ be any subset. Prove the following assertions, which were made in Lemma 6.7 of the lecture:

(a) The commutant S' of S is a weakly (and thus also strongly) closed unital subalgebra of $B(H)$.

(b) If S satisfies $S^* = S$ (i.e., $x \in S \Rightarrow x^* \in S$), then S' is a weakly (and thus also strongly) closed unital $*$-subalgebra of $B(H)$.

(c) We have $S \subseteq S''$ and $S''' = S'$. If $T \subseteq B(H)$ is another subset, then

$$S \subseteq T \quad \Longrightarrow \quad T' \subseteq S'.$$

Exercise 4 (10 points)**.** Let $(H, \langle \cdot, \cdot \rangle)$ be a separable complex Hilbert space.

(a) Let $(\xi_i)_{i\in I}$ and $(\eta_j)_{j\in J}$ two orthonormal bases of H. Show that for all $x \in B(H)$

$$\sum_{i\in I} \|x\xi_i\|^2 = \sum_{j\in J} \|x\eta_j\|^2.$$

(b) We set for $x \in B(H)$

$$\|x\|_2 := \left(\sum_{i\in I} \|x\xi_i\|^2 \right)^{1/2} \in [0, \infty],$$

where $(\xi)_{i\in I}$ is any orthonormal basis of H; we call x a *Hilbert-Schmidt operator* if $\|x\|_2 < \infty$. We denote by $L^2(B(H))$ the set of all Hilbert-Schmidt operators $x \in B(H)$. Prove the following statements for arbitrary $x, y \in B(H)$ and $\lambda \in \mathbb{C}$:

(i) $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$ and $\|\lambda x\|_2 = |\lambda| \|x\|_2$

(ii) $\|x\| \leq \|x\|_2$

(iii) $\|x^*\|_2 = \|x\|_2$

(iv) $\|xy\|_2 \leq \|x\| \|y\|_2$ and $\|xy\|_2 \leq \|x\|_2 \|y\|$

Thus, $L^2(B(H))$ is a selfadjoint ideal in $B(H)$ and a normed $*$-algebra.