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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture ***Operator algebras (Functional analysis II)***
Summer term 2018

Sheet 6
submission: Wednesday, May 23 2018 (!), before the lecture

Exercise 1 (10 points)**.**

(a) Let $(x)_{\lambda\in\Lambda}$ and $(y_\lambda)_{\lambda\in\Lambda}$ be two nets in $B(H)$ over the same directed set Λ which are strongly convergent to x and y, respectively, in $B(H)$. Suppose, moreover, that $(x_\lambda)_{\lambda\in\Lambda}$ is bounded. Prove that $(x_\lambda y_\lambda)_{\lambda\in\Lambda}$ is strongly convergent to xy.

Deduce that, if $h_1, h_2 \in C(\mathbb{R})$ are both SOT continuous and at least one of them is additionally bounded, then also $h_1 \cdot h_2$ is SOT continuous.

(This argument was used in the proof of Theorem 7.2.)

(b) Put $B(H)_1 := \{x \in B(H) \mid \|x\| \leq 1\}$. Show that, if $(x_\lambda)_{\lambda\in\Lambda}$ is any net in $B(H)_1$ which converges strongly to some $x \in B(H)$, then necessarily $x \in B(H)_1$.

(Note that this proves the inclusions "$\subseteq$" in the Kaplansky density theorem, Theorem 7.3.)

Exercise 2 (10 points)**.** Prove Corollary 7.5 of the lecture:

Let $A \subseteq B(H)$ be a $*$-subalgebra. Then, for each $x \in B := \overline{A}^{\mathrm{SOT}}$, we find a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that

$$x_\lambda \xrightarrow{\mathrm{SOT}} x \qquad \text{and} \qquad \sup_{\lambda\in\Lambda} \|x_\lambda\| \leq \|x\|.$$

If x is selfadjoint, then $(x_\lambda)_{\lambda\in\Lambda}$ can be chosen to consist of selfadjoint operators x_λ in A.

If the underlying Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is separable, then there exists even a sequence $(x_n)_{n=1}^\infty$ in A with the above properties.

Hint: For proving the existence of the sequence $(x_n)_{n=1}^\infty$, take any sequence $(\xi_k)_{k=1}^\infty$ of vectors in H for which $\{\xi_n \mid n \in \mathbb{N}\}$ is dense in H and find $x_n \in A$ for each $n \in \mathbb{N}$ such that $\|x_n\| \leq \|x\|$ and $\|x_n\xi_k - x\xi_k\| < \frac{1}{n}$ holds for $k = 1, \ldots, n$. Deduce that $(x_n)_{n=1}^\infty$ converges strongly to x.

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Exercise 3 (5 points)**.** Prove Lemma 8.9 of the lecture:

> An abelian von Neumann algebra $M \subseteq B(H)$ is maximal if and only if $M' = M$.

Exercise 4 (15 points)**.** Let K be a compact Hausdorff space and let μ be a finite Borel measure on K. For each $f \in L^\infty(K, \mu)$, we define on the Hilbert space $L^2(K, \mu)$ a multiplication operator $M_f \in B(L^2(K, \mu))$ by $M_f g := fg$ for all $g \in L^2(K, \mu)$. We denote by $A := \{M_f \mid f \in C(K)\}$ the C^*-algebra of multiplication operators induced by continuous functions. Prove the following statements:

(a) We have $M := A' = \{M_f \mid f \in L^\infty(K, \mu)\}$ and M is a von Neumann algebra.

(b) We have $M' = M$, i.e., M is maximal abelian.

(c) The constant function $\mathbf{1}$ (given by $\mathbf{1}(t) = 1$ for all $t \in K$) is cyclic and separating.