Let $M$ be a von Neumann algebra, $e \in M$ a projection. We say that

(i) $e \neq 0$ is minimal, if for every projection $f \in M$ with $f \leq e$ we have $f = 0$ or $f = e$.

(ii) $e$ is finite if $e \sim f \leq e$ implies $e = f$.

**Exercise 1** (10 points). Consider the von Neumann algebra $M = B(H)$, where $H$ is a separable Hilbert space and let $e, f \in B(H)$ be projections. Show that

(a) $e \sim f$ if and only if $\dim eH = \dim fH$.

(b) $e \not\sim f$ if and only if $\dim eH \leq \dim fH$.

(c) $e$ is minimal if and only if $e$ is a projection of rank 1.

(d) $e$ is finite if and only if $eH$ is finite dimensional.

**Exercise 2** (10 points). Let $M$ be a von Neumann algebra such that 1 is a finite projection. Prove the following statements:

(a) Every projection $e \in M$ is finite.

(b) If $e \sim f$ then $1 - e \sim 1 - f$ for projections $e, f \in M$.

(c) If $e \sim f$ then there exists a unitary $u \in M$ such that $f = ueu^*$. 

(d) The statements (b) and (c) are false if $M$ is a von Neumann algebra, where 1 is not finite.

**Hint:** Consider the unilateral Shift.