Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 8
submission: Monday, June 4 2018, before the lecture

Exercise 1 (10 points). Let $M \subseteq B(H)$ be a factor and let $e, f \in M$ be projections. Show:

(a) If $e$ and $f$ are minimal, then $e \sim f$.

(b) If $f$ is finite and $e \preceq f$, then $e$ is also finite.

(c) It holds $ef = 0$ if and only if $eH \perp fH$. In this case, $e + f$ is a projection.

Exercise 2 (10 points). Let $M \subseteq B(H)$ be a factor and let $e, f \in M$ be projections. Prove that there exist a family $(e_i)_{i \in I}$ of mutually orthogonal projections in $M$ with $e_i \sim e$ for all $i \in I$ and a projection $r \in M$ with $r \preceq e$ and $r \not\sim e$ such that

$$f = r + \sum_{i \in I} e_i.$$ 

Hint: Use Zorn’s lemma.

Exercise 3 (10+10+10* points). Let $M \in B(H)$ be a von Neumann algebra and let $p \in M$ be a non-zero projection. Prove the following statements:

(a) We have $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on the Hilbert space $pH = \text{im}(p)$. Thus $pMp$ and $M'p$ are both von Neumann algebras on $pH$.

Hint: First show that $(pM)' = pMp$ holds. Conclude by proving that any unitary $u \in (pMp)'$ can be extended to an isometry $\tilde{u} : K \rightarrow K$ on the Hilbert space $K := \overline{MpH} \subseteq H$, such that $\tilde{u}q \in M'$ holds for $q$ being the orthogonal projection from $H$ to $K$. For this purpose, check $q \in Z(M)$.

(b) If $M$ is a factor, then $pMp$ and $pM'$ are both factors on $pH$. Moreover, the map

$$\Phi : M' \rightarrow M'p, \quad x \mapsto xp$$

is a weakly continuous $*$-algebra isomorphism.

Hint: Use the general fact (which was proven in (a)) that the orthogonal projection $q$ from $H$ onto $K = \overline{MpH}$ belongs to $Z(M)$.

(c*) If $M$ is a factor of type I, II or III, then $pMp$ is a factor of the same type.