UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG MATHEMATIK
Dr. Tobias Mai
Prof. Dr. Moritz Weber
M.Sc. Simon Schmidt

Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 8
submission: Monday, June 4 2018, before the lecture

Exercise 1 (10 points)**.** Let $M \subseteq B(H)$ be a factor and let $e, f \in M$ be projections. Show:

(a) If e and f are minimal, then $e \sim f$.

(b) If f is finite and $e \precsim f$, then e is also finite.

(c) It holds $ef = 0$ if and only if $eH \perp fH$. In this case, $e + f$ is a projection.

Exercise 2 (10 points)**.** Let $M \subseteq B(H)$ be a factor and let $e, f \in M$ be projections. Prove that there exist a family $(e_i)_{i \in I}$ of mutually orthogonal projections in M with $e_i \sim e$ for all $i \in I$ and a projection $r \in M$ with $r \precsim e$ and $r \nsim e$ such that

$$f = r + \sum_{i \in I} e_i.$$

Hint: Use Zorn's lemma.

Exercise 3 (10+10+10* points)**.** Let $M \in B(H)$ be a von Neumann algebra and let $p \in M$ be a non-zero projection. Prove the following statements:

(a) We have $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on the Hilbert space $pH = \operatorname{im}(p)$. Thus pMp and $M'p$ are both von Neumann algebras on pH.

Hint: First show that $(pM')' = pMp$ holds. Conclude by proving that any unitary $u \in (pMp)'$ can be extended to an isometry $\tilde{u} : K \to K$ on the Hilbert space $K := \overline{MpH} \subseteq H$, such that $\tilde{u}q \in M'$ holds for q being the orthogonal projection from H to K. For this purpose, check $q \in Z(M)$.

(b) If M is a factor, then pMp and pM' are both factors on pH. Moreover, the map

$$\Phi : M' \to M'p, \quad x \mapsto xp$$

is a weakly continuous $*$-algebra isomorphism.

Hint: Use the general fact (which was proven in (a)) that the orthogonoal projection q from H onto $K = \overline{MpH}$ belongs to $Z(M)$.

(c*) If M is a factor of type I, II or III, then pMp is a factor of the same type.