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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 9
submission: Monday, June 11 2018, before the lecture

Exercise 1 (10 points)**.** Prove Remark 10.1 (iii) of the lecture:

> Let H_1, H_2 be Hilbert spaces. If $x \in B(H_1)$ and $y \in B(H_2)$ are given, then
> $$(x \otimes y)(\xi \otimes \eta) := (x\xi) \otimes (y\eta) \qquad \text{for all } \xi \in H_1, \eta \in H_2$$
> defines a linear operator $x \otimes y : H_1 \otimes H_2 \to H_1 \otimes H_2$ that extends uniquely to an operator $x \hat{\otimes} y \in B(H_1 \hat{\otimes} H_2)$ with $\|x \hat{\otimes} y\| = \|x\| \|y\|$.

Hint: Use the universal property of tensor products in order to establish the existence and uniqueness of the linear map $x \otimes y : H_1 \otimes H_2 \to H_1 \otimes H_2$. For proving the statements about its extension to $H_1 \hat{\otimes} H_2$, consider first unitary operators $x \in B(H_1), y \in B(H_2)$ and show that $x \otimes y$ extends to $x \hat{\otimes} y$; deduce that $x \otimes y$ extends to $x \hat{\otimes} y$ for general $x \in B(H_1), y \in B(H_2)$.

Exercise 2 (10 points)**.** Let H_1, H_2 be Hilbert spaces. Show that, if M_i is a factor on H_i $(i = 1, 2)$, then the von Neumann algebra tensor product $M_1 \overline{\otimes} M_2$ is also a factor.

Exercise 3 (20 points)**.** Let $M \subseteq B(H)$ be a finite dimensional von Neumann algebra.

(a) Prove that pMp is a factor on pH for each minimal projection p in the center $Z(M)$.

(b) Show that the center $Z(M)$ is a finite dimensional abelian von Neumann algebra, which can be written as
$$Z(M) = \bigoplus_{i=1}^{l} (\mathbb{C} p_i),$$
where $p_1, \ldots, p_l$ denote the minimal projections in $Z(M)$.

(c) Deduce that there are $l \in \mathbb{N}$ and $n_1, \ldots, n_l \in \mathbb{N}$, such that M is $*$-isomorphic to $\bigoplus_{i=1}^{l} M_{n_i}(\mathbb{C})$. In fact, there are complex Hilbert spaces $K_1, \ldots, K_l$ and a unitary $U : H \to \bigoplus_{i=1}^{l} (K_i \otimes \mathbb{C}^{n_i})$, such that
$$UMU^* = \bigoplus_{i=1}^{l} (\mathrm{id}_{K_i} \otimes B(\mathbb{C}^{n_i})).$$