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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 10
submission: Monday, June 18 2018, before the lecture

Exercise 1 (10 points)**.** Let $L(\mathbb{Z})$ be the left group von Neumann algebra of the discrete group $(\mathbb{Z}, +)$.

(a) Show that $L(\mathbb{Z})$ is an abelian von Neumann algebra.

(b) Prove that $L(\mathbb{Z})$ is $*$-isomorphic to $L^\infty(\mathbb{T}, m)$, where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the unit circle and m the arc length measure on $\mathbb{T}$. Furthermore, show that the tracial state $\tau : L(\mathbb{Z}) \to \mathbb{C}, x \mapsto \langle x\delta_e, \delta_e\rangle$ corresponds under that isomorphism to the linear functional on $L^\infty(\mathbb{T}, m)$ that is given by $f \mapsto \int_{\mathbb{T}} f(\zeta)\, dm(\zeta)$.

Exercise 2 (20 points)**.** Consider the chain of inclusions

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow M_{2^3}(\mathbb{C}) \hookrightarrow \ldots \hookrightarrow M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C}) \hookrightarrow \ldots$$

which are given by

$$\iota_n : \; M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C}), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

(a) Justify that its union $A := \bigcup_{n=1}^\infty M_{2^n}(\mathbb{C})$ is a complex unital $*$-algebra and show that there exists a (well-defined!) linear functional $\tau_0 : A \to \mathbb{C}$, such that $\tau_0(x) = \mathrm{tr}_{2^n}(x)$ holds for every $x \in M_{2^n}(\mathbb{C})$, where tr_{2^n} denotes the normalized trace on $M_{2^n}(\mathbb{C})$. Deduce that τ_0 is unital, positive, faithful, and tracial.

(b) Denote by H the Hilbert space which is obtained by completion of A with respect to the inner product given by $\langle x, y\rangle = \tau_0(xy^*)$. Prove that each $y \in A$ induces a bounded linear operator on H, i.e., we can view $A \subseteq B(H)$.

(c) Consider the von Neumann algebra $\mathcal{R} := A'' \subseteq B(H)$. Show that there exists a unique faithful normal tracial state τ on $\mathcal{R}$.

(d) Prove that $\mathcal{R} \subseteq B(H)$ is a factor of type II_1.

Hint: Since the center $Z(\mathcal{R}) = \mathcal{R} \cap \mathcal{R}'$ of $\mathcal{R}$ is generated by its positive elements, factoriality follows as soon as we have shown that any positive $z \in Z(\mathcal{R})$ is a positive multiple of 1. For doing so, use the result obtained in (c).

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Exercise 3 (10 points)**.**

(a) Prove Remark 11.11 (ii) from the lecture:

> Let $M \subseteq B(H)$ be a von Neumann algebra that has a faithful tracial state $\tau : M \to \mathbb{C}$ which is moreover normal. Then M is finite.

Is this implication still true if τ is not required to be normal?

(b) Let $M \subseteq B(H)$ be a factor and let τ be a faithful tracial state on M. Consider any two projections $e, f \in M$. Show that $e \sim f$ if and only if $\tau(e) = \tau(f)$.