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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture ***Operator algebras (Functional analysis II)***
Summer term 2018

Sheet 11
submission: Monday, June 25 2018, before the lecture

Exercise 1 (10 points)**.** Let M be a factor of type II_1 and let $\tau : M \to \mathbb{C}$ be its unique faithful normal tracial state. Show that

$$\tau(\mathcal{P}(M)) = [0,1].$$

Hint: Fix any $t \in [0,1]$ and consider the set $S_t := \{p \in \mathcal{P}(M) \mid \tau(p) \leq t\}$. Verify that S_t is partially ordered and use Zorn's lemma to prove that S_t contains a maximal element p; finally, show that $\tau(p) = t$.

Exercise 2 (10 points)**.** Let $M \subseteq B(H)$ be a type II_1 factor with its unique faithful normal tracial state $\tau : M \to \mathbb{C}$. Suppose that M possesses a cyclic and separating vector Ω in H such that $\tau(x) = \langle x\Omega, \Omega\rangle$ for all $x \in M$. Denote by M' the commutant of M in $B(H)$ and let $J : M\Omega \to M\Omega$ be defined by $J(x\Omega) = x^*\Omega$ for all $x \in M$. Prove the following statements:

(a) The antilinear operator $J : M\Omega \to M\Omega$ extends uniquely to an antilinear isometry $J : H \to H$ that satisfies $J^2 = 1$ and $\langle J\xi, \eta\rangle = \langle J\eta, \xi\rangle$ for all $\xi, \eta \in H$; we call J the *canonical conjugation operator on H*.

(b) For all $x, y \in M$, it holds true that $JxJ(y\Omega) = yx^*\Omega$.

(c) For every $x \in M'$, we have that $Jx\Omega = x^*\Omega$.

Deduce that $JMJ = M'$ and show that also M' is a type II_1 factor. How does the unique faithful tracial state on M' look like?

Hint: For proving $JMJ = M'$, switch the roles of M and M'. What does (c) tell us about this case?

Exercise 3 (10 points)**.** Let G be a non-trivial countable discrete i.c.c. group. Consider the left group von Neumann algebra $M = L(G)$, which is a factor of type II_1, with its unique faithful normal tracial state τ.

(a) Show that $L^2(M,\tau)$ and $\ell^2(G)$ are isomorphic as (left) M-modules.

(b) Conclude that also the right group von Neumann algebra $R(G)$ is a type II_1 factor.

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Exercise 4 (10 points)**.** Let M be a separable factor of type II_1 and let $(H_i)_{i\in I}$ be a countable family of separable M-modules. Prove that

$$\dim_M\left(\bigoplus_{i\in I} H_i\right) = \sum_{i\in I} \dim_M H_i.$$

Hint: Show that $\bigoplus_{i\in I}(L^2(M,\tau) \mathbin{\hat{\otimes}} \ell^2(\mathbb{N}))$ and $L^2(M,\tau) \mathbin{\hat{\otimes}} \ell^2(\mathbb{N})$ are equivalent M-modules, where τ denotes the unique faithful normal tracial state on M.

Exercise 5 (10^* points)**.** Consider the group $U_n(\mathbb{C})$ of unitary matrices in $M_n(\mathbb{C})$. If $U_n(\mathbb{C})$ is endowed with the topology induced by the restriction of the operator norm on $M_N(\mathbb{C}) = B(\mathbb{C}^n)$, it forms a compact group. Let μ be a Radon probability measure defined on the Borel sets of $U_n(\mathbb{C})$ which is *(left) invariant* in the sense that $\mu(u\Omega) = \mu(\Omega)$ holds for all Borel subsets $\Omega \subseteq U_n(\mathbb{C})$ and all $u \in U_n(\mathbb{C})$. (Note that such a measure exists and is in fact unique; it is called the *(left) Haar measure of* $U_n(\mathbb{C})$.) Show that for all $x \in M_n(\mathbb{C})$

$$\mathrm{tr}_n(x)1 = \int_{U_n(\mathbb{C})} uxu^*\, d\mu(u).$$

Hint: Put $y := \int_{U_n(\mathbb{C})} uxu^*\, d\mu(u) \in M_n(\mathbb{C})$. Show that $y \in Z(M_n(\mathbb{C}))$ by proving that $y \in U_n(\mathbb{C})'$. Finally, use these observations to compute $\mathrm{tr}_n(y)$.