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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 12
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Exercise 1 (20 points)**.**

(a) Show that the universal C^*-algebras

(i) $C^*(e_{ij}, i,j = 1,\dots,n \,|\, e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il})$, $C^*(x_1,\dots,x_n \,|\, x_i^* x_j = \delta_{ij}x_1)$

(ii) $C^*(e_{ij}, i,j \in \mathbb{N} \,|\, e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il})$, $C^*(x_i, i \in \mathbb{N} \,|\, x_i^* x_j = \delta_{ij}x_1)$

do exist (i.e. that their C^*-seminorms are bounded).

(b) Find *-homomorphisms

$$\varphi : C^*(e_{ij}, i,j = 1,\dots,n \,|\, e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il}) \to C^*(x_1,\dots,x_n \,|\, x_i^* x_j = \delta_{ij}x_1),$$
$$\psi : C^*(x_1,\dots,x_n \,|\, x_i^* x_j = \delta_{ij}x_1) \to C^*(e_{ij}, i,j = 1,\dots,n \,|\, e_{ij}^* = e_{ji}, e_{ij}e_{kl} = \delta_{jk}e_{il})$$

such that $\varphi \circ \psi = \mathrm{id}_{C^*(x_1,\dots,x_n \,|\, \dots)}$ and $\psi \circ \varphi = \mathrm{id}_{C^*(e_{ij}, i,j=1,\dots,n \,|\, \dots)}$. Show that both universal C^*-algebras are isomorphic to $M_n(\mathbb{C})$.

(c) Similar to (b), find *-homomorphisms between the C^*-algebras in (a), (ii). Show that both universal C^*-algebras are isomorphic to $\mathcal{K}(H)$, where H is an infinite dimensional, separable Hilbert space.

(d) Show that the universal C^*-algebra $C^*(x \,|\, x^2 = 0, x = xx^*x)$ exists and that it is isomorphic to $M_2(\mathbb{C})$.

Hint: The element x is a partial isometry, so x^*x corresponds to a rank 1 projection in $M_2(\mathbb{C})$.

Exercise 2 (10 points)**.**

(a) Find $n \in \mathbb{N}$ such that $C^*(p, 1 \,|\, p \text{ projection}) \cong \mathbb{C}^n$ and prove the isomorphism. (Here 1 is the unit of the C^*-algebra, so the relations $1p = p1 = p$ hold as well.)

(b) Find $m \in \mathbb{N}$ such that $C^*(s, 1 \,|\, s \text{ symmetry}) \cong \mathbb{C}^m$ and prove the isomorphism. A *symmetry* is a selfadjoint unitary.

(c) Give an explicit isomorphism between $C^*(p, 1 \,|\, p \text{ projection})$ and $C^*(s, 1 \,|\, s \text{ symmetry})$, if $m = n$. If $m \neq n$, think about (a) and (b) again.

Exercise 3 (10 points)**.** Let H be an infinite dimensional, separable Hilbert space. Show that every ideal I in $B(H)$ contains the compact operators $\mathcal{K}(H)$. In order to do this, verify first that all rank 1 operators are contained in I. Deduce that $\mathcal{K}(H)$ is simple.