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FACHRICHTUNG MATHEMATIK
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Exercises for the lecture *Operator algebras (Functional analysis II)*
Summer term 2018

Sheet 13
submission: Monday, July 9 2018, before the lecture

Exercise 1 (20 points)**.** Let $\vartheta = \frac{p}{q} \in \mathbb{Q}$.

(a) Find a representation $\pi : A_\vartheta \to M_q(\mathbb{C})$.

(b) Prove that there exist unital *-homomorphisms $\varphi : A_\vartheta \to B$ and $\psi : A_\vartheta \to D$ such that $\varphi(v^q) = 1$ and $\psi(v^q) \neq 1$.

(c) Deduce that A_ϑ is not simple.

(d) Show that u^q and v^q commute with every element in A_ϑ, where u, v denote the generators of A_ϑ. Thus, there is a *-homomorphism $C(\mathbb{T}^2) \to C^*(u^q, v^q) \subseteq A_\vartheta$ (in fact, this is an isomorphism).

Exercise 2 (20 points)**.** Consider $\mathcal{O}_n := C^*(S_1, \ldots, S_n \mid S_i^* S_i = 1, \sum_{i=1}^n S_i S_i^* = 1)$, the Cuntz algebra for $2 \leq n \in \mathbb{N}$. A *word* in $\mathcal{O}_n$ is an element $S_\mu = S_{i_1} \ldots S_{i_k}$, where $\mu = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$. Here $|\mu| := k$ denotes the length of the word S_μ and μ is called multi-index.

(a) Prove the following statements.

(i) If $|\mu| = |\nu|$, then $S_\mu^* S_\nu = \delta_{\mu\nu} 1$.

(ii) If $|\mu| < |\nu|$, then $S_\mu^* S_\nu = \begin{cases} S_{\nu'} & \text{if } \nu = \mu\nu' \\ 0 & \text{else.} \end{cases}$

(iii) If $|\mu| > |\nu|$, then $S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \\ 0 & \text{else.} \end{cases}$

(b) Let $k \in \mathbb{N}$. Show that $\sum_{\delta \text{ multi-index}, |\delta| = k} S_\delta S_\delta^* = 1$.

(c) Let μ, ν be such that $|\mu| \neq |\nu|$ and $|\mu|, |\nu| \leq k$. Verify $S_\gamma^*(S_\mu S_\nu^*) S_\gamma = 0$ for $S_\gamma = S_1^k S_2$.

(d) Let $k \in \mathbb{N}$. Show:

(i) $w := \sum_{|\delta| = k} S_\delta S_\gamma S_\delta^* \in \mathcal{O}_n$ is an isometry, with $S_\gamma := S_1^{2k} S_2$.

(ii) $wx = xw$ for all $x \in \text{span}\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k\}$.

(iii) $w^* S_\mu S_\nu^* w = \begin{cases} S_\mu S_\nu^* & \text{if } |\mu| = |\nu| \\ 0 & \text{else} \end{cases}$ for all $|\mu|, |\nu| \leq k$.