JSF-1

Subfactors [Introduction to subfactors, Jones-Sunder] Jonas, SB, Jan 2016

4.1.2016

0 Introduction

topics in von Neumann algebras: (i) free probability (Voiculescu)
(ii) deformation/rigidity (of Popa-Vaes)
(iii) subfactors (Jones)

$$\begin{tikzcd}
(i) \arrow[rr, leftrightarrow, "\text{Classification of vN alg}"] \arrow[dr, leftrightarrow, "\text{partitions \& combinatorics}"'] & & (ii) \arrow[dl, leftrightarrow, "\text{tool: Hilbert bimodules}"] \\
& (iii) &
\end{tikzcd}$$

1 Basics on bimodules

1.1 **Def:** $M \subseteq B(H)$ von Neumann algebra.

(i) M factor $:\Longleftrightarrow$ $M \cap M' = \mathbb{C}1$ $\left(\begin{array}{l}\tau \text{ continuous in WOT} \\ \Rightarrow \tau \text{ normal}\end{array}\right)$

(ii) M finite $:\Longleftrightarrow$ $\exists\, \tau : M \to \mathbb{C}$ normal faithful trace state

$\uparrow$
$\left[\begin{array}{l} (p_i)_{i\in I} \text{ projections, increasing, bounded} \\ \Rightarrow \tau(\sup p_i) = \sup \tau(p_i) \end{array} \right]$

(iii) M II_1-factor $:\Longleftrightarrow$ M finite, infinite-dimensional factor

Example: $H \subseteq G$ icc groups $\Rightarrow$ $L(H) \subseteq L(G)$

$$[G:H] := \left| G/H \right| = \left| \{ gH \mid g \in G \} \right|$$

What about an index $[LG : LH]$?

Reminder: (M,τ) II_1-factor, $\xrightarrow{\text{GNS}}$ $L^2(M)$ Hilbert space with $\langle x,y\rangle := \tau(y^*x)$

M acts on $L^2(M)$ by left multiplication $x\hat{y} = \widehat{xy}$ for $x,y \in M$ ($\hat{x} \in L^2(M)$)

$\Rightarrow$ $\exists\, \pi_\ell : M \to B(L^2(M))$ *-hom. with $\pi_\ell(x)\hat{y} := \widehat{xy}$

and $\exists\, \pi_r : M \to B(L^2(M))$ *-antihom. with $\pi_r(x)\hat{y} := \widehat{yx}$

$(\uparrow\, \pi_r(xy) = \pi_r(y)\pi_r(x))$

We see that π_ℓ and π_r commute.

The map $J : \begin{array}{c} M \to M \\ x \mapsto x^* \end{array}$ extends to a map $\begin{array}{c} L^2(M) \to L^2(M) \\ \hat{x} \mapsto \widehat{x^*} \end{array}$

and we have $\pi_r(x) = J\pi_\ell(x^*)J$, $x \in M$

1.2 **Def:** M, N vNAlg.

(v) An M-N-bimodule is irreducible if ${}_M B_M(H) = \mathbb{C}1$.

(i) Left M-Module: H Hilbert space, $\pi_\ell : M \to B(H)$ unital, normal $*$-hom. Write ${}_M H$ and $x\xi := \pi_\ell(x)\xi$, $x \in M, \xi \in H$.

(ii) Right M-Module: H Hilbert space, $\pi_r : M \to B(H)$ unital, normal $*$-antihom. Write H_M and $x\xi := \pi_r(x)\xi$, $x \in M, \xi \in H$.

(iii) M-N-Bimodule: H Hilbert space, left M-module, right N-module such that π_ℓ, π_r commute.

(iv) Left modular operator on ${}_M H$: $T \in B(H)$ such that $Tx \underset{\uparrow}{=} xT$ $\forall x \in M$.
Write ${}_M B(H)$ for all left modular operators. (i.e. T and π_ℓ commute)
(or $T \in B({}_M H, {}_M K)$) (analogous: right / bimodular operator)

Remark: ${}_M H_K \underset{\uparrow}{\cong} {}_M L_K$ as left module and right module $\not\Rightarrow$ $\cong$ as bimodule
$\exists\, U : H \to L$ unitary s.t. actions fit

Classification of M-modules

Example: M finite vN algebra $\Rightarrow$ ${}_M L^2(M)$ left M-module.

H Hilbert space $\Rightarrow$ ${}_M(L^2(M) \otimes H)$ left M-module via $x(\xi \otimes \eta) := (x\xi) \otimes \eta$, $x \in M, \xi \in L^2(M), \eta \in H$.

${}_M K$ left M-module, $p \in {}_M B(K)$ projection $\Rightarrow$ ${}_M(pK)$ left M-module via $xp\xi := \pi_\ell(x)\xi$

1.3 Lemma: M II_1-factor (separable), ${}_M H$ separable M-module
$\Rightarrow \exists\, p \in {}_M B(L^2(M) \otimes \ell^2(\mathbb{N}))$ projection: ${}_M H \underset{\uparrow \text{ as } M\text{-modules}}{\cong} {}_M p(L^2(M) \otimes \ell^2(\mathbb{N}))$
(i.e. $\exists\, u \in {}_M B(H, p(L^2(M) \otimes \ell^2(\mathbb{N})))$ unitary)

This defines a bijection between isomorphism classes of left M-modules and equivalence classes of projections in ${}_M B(L^2(M) \otimes \ell^2(\mathbb{N}))$

(i.e. ${}_M p(L^2(M) \otimes \ell^2(\mathbb{N})) \underset{\uparrow \text{ as } M\text{-modules}}{\cong} {}_M q(L^2(M) \otimes \ell^2(\mathbb{N})) \Leftrightarrow p \underset{\uparrow \text{ MvN}}{\sim} q$ $\forall p, q \in {}_M B(L^2(M) \otimes \ell^2(\mathbb{N}))$ projections
(in ${}_M B(L^2(M) \otimes \ell^2(\mathbb{N}))$))

Remark: (i) ${}_{\mathcal{M}}\mathcal{B}(L^2(\mathcal{M})\otimes \ell^2(\mathbb{N})) = (\mathcal{M}\otimes 1)' = \mathcal{M}'\otimes \mathcal{B}(\ell^2(\mathbb{N}))$ II_∞-factor (if $\mathcal{M}$ II_1-factor)

(ii) $(e_n)_{n\in\mathbb{N}}$ ONB of $\ell^2(\mathbb{N})$. $Tr: \mathcal{B}(\ell^2(\mathbb{N}))_+ \to [0,\infty]$ is a trace
$x \longmapsto \sum_{n\in\mathbb{N}} \langle xe_n, e_n\rangle$

weight on $\mathcal{B}(\ell^2(\mathbb{N}))$ with

Def of a trace weight

(a) $Tr(x+y) = Tr(x) + Tr(y)$ $\quad x,y\in \mathcal{B}(\ell^2(\mathbb{N}))_+$

(b) $Tr(\lambda x) = \lambda Tr(x)$ $\quad , \lambda \geq 0, x\in \mathcal{B}(\ell^2(\mathbb{N}))_+$

(c) $Tr(x^* x) = Tr(xx^*)$ $\quad , x\in \mathcal{B}(\ell^2(\mathbb{N}))$

(d) semifinite: $\forall 0\neq x\in \mathcal{B}(\ell^2(\mathbb{N}))_+ \; \exists 0\neq y \in \mathcal{B}(\ell^2(\mathbb{N}))_+ : y\leq x, Tr(y) < \infty$

(e) Tr normal

(f) faithful: $Tr(x) = 0 \Leftrightarrow x = 0 \quad \forall x\in \mathcal{B}(\ell^2(\mathbb{N}))_+$

1.4 Def: $\mathcal{M}$ II_1-factor, ${}_{\mathcal{M}}H$ separable $\mathcal{M}$-module. **Dimension** of ${}_{\mathcal{M}}H$:

$\dim_{\mathcal{M}-} H := (\tau_{\mathcal{M}'} \otimes Tr)(p)$ for $p\in \mathcal{M}'\otimes \mathcal{B}(\ell^2(\mathbb{N}))$ projection

with ${}_{\mathcal{M}}H \cong p(L^2(\mathcal{M})\otimes \ell^2(\mathbb{N}))$

well defined since $p\sim q \Leftrightarrow (\tau_{\mathcal{M}'}\otimes Tr)(p) = (\tau_{\mathcal{M}'}\otimes Tr)(q)$

Facts: • left $\mathcal{M}$-modules classified by dimension

- $\mathcal{M}$ II_1-factor, ${}_{\mathcal{M}}H$ left $\mathcal{M}$-module. $\dim_{\mathcal{M}-}H < \infty \Leftrightarrow \pi_\ell(\mathcal{M})'$ II_1-factor
- $\dim_{\mathcal{M}-} H < \infty$, $p' \in \pi_\ell(\mathcal{M})'$ projection $\Rightarrow \dim_{\mathcal{M}-}(p'H) = \tau_{\pi_\ell(\mathcal{M})'}(p')\dim_{\mathcal{M}-} H$
- $\pi_\ell : \mathcal{M} \hookrightarrow \mathcal{B}(H)$ faithful, $\dim_{\mathcal{M}-} H < \infty \Rightarrow \dim_{\pi_\ell(\mathcal{M})'_-}(H) = (\dim_{\mathcal{M}} H)^{-1}$
- $p\in\mathcal{M}$ projection $\Rightarrow \dim_{p\mathcal{M}p}(pH) = \dfrac{\dim_{\mathcal{M}} H}{\tau(p)}$

Example: $\dim_{\mathcal{M}-} L^2(\mathcal{M}) = 1$, since for $p_1 \in \mathcal{B}(\ell^2(\mathbb{N}_0))$ proj. on $\langle e_1\rangle$,

we have ${}_{\mathcal{M}}L^2(\mathcal{M}) \cong (1\otimes p_1)(L^2(\mathcal{M})\otimes \ell^2(\mathbb{N}))$

and $\tau_{\mathcal{M}'}\otimes Tr(1\otimes p_1) = 1$

Remark: Bimodules will correspond to the representation theory of subfactors.

2 Index for subfactors [Jones, 83]

6.1.2016

M II_1-factor. Then $\dim_M$ full invariant for left M-modules.

In particular: ${}_M H_M$, ${}_M K_M$ M-bimodules. ${}_M H_M \underset{\uparrow \text{as bimodules}}{\cong} {}_M K_M \overset{\S 1}{\underset{\not\Leftarrow}{\Rightarrow}} \dim_{M-} H = \dim_{M-} K$ & $\dim_{-M} H = \dim_{-M} K$

„$\not\Leftarrow$": $\vartheta : M \to M$ outer isomorphism (inner: $\vartheta(x) = uxu^*$, $u \in M$ unitary)

$H_\vartheta := {}_{\vartheta(M)} L^2(M)_M$ with usual right action, but left action $x\hat{y} = \widehat{\vartheta(x)y}$, $x, y \in M$

Then $\dim_{M-} H_\vartheta = \dim_{-M} H_\vartheta = 1$, but many different H_ϑ's are possible.

2.1 **Def:** M II_1-factor, $N \subseteq M$ subfactor, if N is a von Neumann subalgebra which is also a factor with $1_M \in N$.

$N \subseteq M$ irreducible, if $N' \cap M = \mathbb{C}1$.

2.2 **Remark:** ${}_M H_M$ bimodule. Then $\pi_l(M) \subseteq \pi_r(M)'$. If $\dim_{M-} H < \infty$, this is a subfactor. And ${}_M H_M$ is irreducible if and only if the subfactor is irreducible.

2.3 **Example:** (a) $H \leq G$ icc $\Rightarrow$ $L(H) \subseteq L(G)$ subfactor

(b) N II_1-factor $\Rightarrow$ $\underset{x \mapsto \mathrm{id} \otimes x}{N \subseteq M_n(\mathbb{C}) \otimes N}$ subfactor

(c) G finite group, N II_1-factor, $G \curvearrowright N$ $\Rightarrow$ $N \subseteq N \rtimes G$ subfactor

2.4 **Def:** $N \subseteq M$ subfactor. $[M:N] := \dim_{N-} L^2(M) \in [1, \infty]$ $\leftarrow$ since $L^2(N) \subseteq L^2(M)$

„how often does ${}_N L^2(N)$ sit inside ${}_N L^2(M)$?"

2.5 **Prop:** ${}_M H$ left M-module with $\dim_{M-} H < \infty$, M II_1-factor, $N \subseteq M$ subfactor, $[M:N] < \infty$.

Then $\dim_{N-} H = [M:N] \dim_{M-} H$, i.e. $[M:N] = \frac{\dim_{N-} H}{\dim_{M-} H}$

$\ulcorner$**Proof:** 1.) $\exists p \in M' \otimes M_n(\mathbb{C})$ projection: ${}_M H \cong p(L^2(M) \otimes \mathbb{C}^n)$ for some $n \in \mathbb{N}$

ΠProof of 1.): Let $n \in \mathbb{N}$ with $\dim_{M-} H \leq n$. Let $p_1 \in M' \otimes B(\ell^2(\mathbb{N}))$ with $\tau_{M'} \otimes \mathrm{Tr}(p_1) = \dim_{M-} H$. Let $q \in B(\ell^2(\mathbb{N}))$ be the projection on $\langle e_1, \dots e_n \rangle \subseteq \ell^2(\mathbb{N})$.

1.2) $\exists p_2 \in M'$ proj. with $\tau_{M'}(p_2) = \frac{\dim_{M-} H}{n} \leq 1$

$\vert\vert\vert$Proof of 1.2): M' II_1-factor and $\tau_{M'} : \mathrm{Proj.}(M')/\sim \to [0,1]$ bij.

Now, $(\tau_{M'} \otimes \mathrm{Tr})(p_2 \otimes q) = \tau_{M'}(p_2) \underbrace{\mathrm{Tr}(q)}_{=n} = \dim_{M-} H = (\tau_{M'} \otimes \mathrm{Tr})(p_1)$

$\Rightarrow p_1 \sim p_2 \otimes q \Rightarrow {}_M H \cong p_1(L^2(M) \otimes \ell^2(\mathbb{N})) \cong (p_2 \otimes q)(L^2(M) \otimes \ell^2(\mathbb{N}))$
$\cong (p_2 \otimes 1)(L^2(M) \otimes \mathbb{C}^n)$

$\llcorner$

Since $p_2 \in M'$ of 1.2), we have $p_2 \in M' \subseteq N'$.

$$\Rightarrow \dim_{N-} H = \dim_{N-}((p_2 \otimes 1)(L^2(M) \otimes \mathbb{C}^n)) \overset{\text{Facts}}{=} \tau_{N'}(p_2) \dim_{N-}(L^2(M) \otimes \mathbb{C}^n) = \overbrace{\tau_{N'}(p_2)}^{\tau_{M'}(p_2) \overset{1.2}{=} \frac{\dim_{M-} H}{n}} \cdot n \cdot \overbrace{\dim_{N-}(L^2(M))}^{[M:N]}$$

⌊

2.7 Example: $H \subseteq G$ icc. Then $[L(G):L(H)] = [G:H]$ (Exercise)

2.6 Corollary: (a) $N \subseteq M$ subfactor, ${}_M H$ left M-module, $\dim_{M-} H < \infty$, $[M:N] < \infty$.
$$\Rightarrow [M:N] = [N':M']$$

(b) $N \subseteq P \subseteq M$ subfactors $\overset{[M:N]<\infty}{\Rightarrow} [M:N] = [M:P][P:N]$

⌈Proof: (a) $[M:N] = \frac{\dim_{N-} H}{\dim_{M-} H} = \frac{(\dim_{M'-} H)^{-1}}{(\dim_{N'-} H)^{-1}} \overset{\text{Facts}}{=} \frac{\dim_{M'-} H}{\dim_{N'-} H} = [N':M']$

(b) $[M:N] = \frac{\dim_{N-} L^2(M)}{\dim_{M-} L^2(M)} = \frac{\dim_{N-} L^2(M)}{\dim_{P-} L^2(M)} \frac{\dim_{P-} L^2(M)}{\dim_{M-} L^2(M)} = [P:N][M:P]$

⌊

2.8 Remark: ${}_M H_M$ M-bimodule, $\dim_{M-} H < \infty$, $\dim_{-M} H < \infty$

$$\Rightarrow \dim_{M-} H \cdot \dim_{-M} H = \dim_{\pi_\ell(M)-} H \cdot \dim_{\pi_r(M)-} H = \frac{\dim_{\pi_\ell(M)-} H}{\dim_{\pi_r(M)'} H} = [\pi_r(M)' : \pi_\ell(M)]$$

Thus, dimensions and indices are linked.

2.9 Theorem: (a) $N \subseteq M$ subfactor $\Rightarrow [M:N] \in \{4\cos^2 \frac{\pi}{n} \mid n \in \mathbb{N}, n \geq 3\} \cup [4,\infty]$

(b) $\forall \lambda \in \{4\cos^2 \frac{\pi}{n} | ..\} \cup [4,\infty]$ $\exists N \subseteq M$ subfactor with $[M:N] = \lambda$.

$\uparrow$

11.1.2016

(c) ⌈for $\lambda \in [4,\infty]$ exists a subfactor R_λ of the hyperfinite factor R
⌊ with $[R:R_\lambda] = \lambda$.

2.10 Lemma: $N \subseteq M$ subfactor, $M \subseteq B(H)$, $\dim H < \infty$, $p \in N' \cap M$ proj.
$$\Rightarrow [pMp : pNp] = tr_{N'}(p)\, tr_M(p)\, [M:N]$$

⌈Proof: $[pMp : pNp] = \frac{\dim_{Np}\, pH}{\dim_{pMp}\, pH} = \frac{\dim_{N-}\, pH}{\dim_{pMp}\, pH} \overset{\text{Facts}}{=} \frac{tr_{N'}(p) \dim_{N-} H}{tr_M(p)^{-1} \dim_{M-} H} = tr_M(p)\, tr_{N'}(p)\, [M:N]$

2.11 Lemma: M II_1-factor, $p \in M$ proj with $pMp \xrightarrow[\vartheta]{\cong} (1-p)M(1-p)$,
$N = \{x + \vartheta(x) \mid x \in pMp\} \Rightarrow N \subseteq M$ subfactor with $[M:N] = \frac{1}{tr_M(p)} + \frac{1}{1 - tr_M(p)}$

⌈Proof: 1) N is a factor, since $N' \cap N \subseteq \{x + \vartheta(x) \mid x \in Z(pMp)\}$

⌈⌈Proof of 1): Let $\tilde{y} + \vartheta(\tilde{y}) \in N' \cap N$ with $\tilde{y} \in pMp$. Show: $\tilde{y} \in (pMp)'$.

For $x \in pMp$: $(\tilde{y} + \vartheta(\tilde{y}))(x + \vartheta(x)) = \tilde{y}x + \vartheta(\tilde{y}x)$
$= (x + \vartheta(x))(\tilde{y} + \vartheta(\tilde{y})) = x\tilde{y} + \vartheta(x\tilde{y})$ $\Rightarrow x\tilde{y} = \tilde{y}x$

⌊⌊

Now, $pMp = N_p$, i.e. $1 = [pMp : Np] \overset{2.10}{=} tr_N(p)\, tr_M(p)\, [M:N]$
and $1 = tr_N(1-p)\, tr_M(1-p)[M:N]$.

└ $\Rightarrow \frac{1}{tr_M(p)} + \frac{1}{1-tr_M(p)} = tr_N(p)[M:N] + tr_N(1-p)[M:N] = [M:N]$

Example: $M = M_2(\mathbb{C})$, $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{C} \right\} \subseteq M$,
$[M:N] = \frac{1}{\frac{1}{2}} + \frac{1}{1-\frac{1}{2}} = 4$ (clear: $\mathbb{C}$ is four times in $M_2(\mathbb{C})$)

Fact: R hyperfinite factor, $p \in R$ proj. $\Rightarrow$ $pRp \cong R$.

┌ Proof of Thm 2.9(c): For $\lambda \in [4, \infty)$, take $t \in (0,1)$ such that $\frac{1}{t} + \frac{1}{1-t} = \lambda$. Since $tr_R(\text{Proj. in } R) = [0,1]$, we find a
└ projection $p \in R$ with $tr_R(p) = t$, then use Lemma 2.11.

3 The basic construction

$N \subseteq M$ subfactor $\Rightarrow L^2(N) \subseteq L^2(M)$.

Let $e_N \in B(L^2(M))$ be the projection onto $L^2(N) \subseteq L^2(M)$. „Jones projection"

Then $e_N(\hat{M}) \subseteq \hat{N}$, where $\hat{M} \subseteq L^2(M)$, such that $E_N : M \to N$ is
"$M\hat{1}_M = M\Omega$", for Ω from the GNS construction

a conditional expectation:
- E_N is completely positive and unital ($id_n \otimes E_N : M_n(\mathbb{C}) \otimes M \to M_n(\mathbb{C}) \otimes N$ positive $\forall n \in \mathbb{N}$)
- E_N N-N-bimodular (hence $E_N^2 = E_N$) ($E_N(n_1 x n_2) = n_1 E_N(x) n_2$, $n_1, n_2 \in N$, $x \in M$)
- E_N trace-preserving ($\tau_N \circ E_N = \tau_M$)
- $E_N(x) e_N = e_N x e_N$
- e_N commutes with $J : L^2(M) \to L^2(M)$, $\hat{x} \mapsto \widehat{x^*}$

3.1 Def: $M_1 := \langle M, e_N \rangle \subseteq B(L^2(M))$ „the basic construction" of $N \subseteq M$.

3.2 Lemma: (i) $e_N \in N'$, (ii) $N = M \cap \{e_N\}'$, (iii) $\langle M, e_N \rangle = JN'J$,
(iv) $\langle M, e_N \rangle$ is a II_1-factor if and only if $[M:N] < \infty$.
Then (a) $[M_1 : M] = [M:N]$, (b) $\tau_{M_1}(e_N) = \frac{1}{[M:N]}$,
(c) $E_M(e_N) = \tau_{M_1}(e_N) 1_M$, where $E_M : M_1 \to M$ „Markov property"

3.3 Corollary: $N \subseteq M_0$ subfactor, $[M:N] < \infty$.

Consider $N \subseteq M_0 \overset{e_1}{\subseteq} M_1 \overset{e_2}{\subseteq} M_2 \subseteq \dots$ with $e_n : L^2(M_{n-1}) \to L^2(M_{n-2})$.

Then (i) $e_n \in M_{n-2}' \cap M_n$, i.e. $e_n e_m = e_m e_n$ for $|n-m| > 1$

(ii) $e_{n+1} e_n e_{n+1} = [M:N]^{-1} e_{n+1}$

(iii) $e_n e_{n+1} e_n = [M:N]^{-1} e_n$

Remark: $e_n \hat{=} |||\underset{n\ \ n+1}{\overset{\sqcup}{\frown}}|||$ as partitions

4 Graphs and index

13.1.

$N \subseteq M$ a subfactor, $[M:N] < \infty$, $N \subseteq M = M_0 \overset{e_1}{\subseteq} M_1 \subseteq M_2 \subseteq \dots$

Consider the Hilbert spaces $L^2(M_n)$, $n \geq 0$ as M-M (or N-N, M-N, N-M) modules to show

$$ {}_N B_N (L^2(M_n)) = N' \cap M_{2n+1}, \quad n \geq 0$$
$$ {}_N B_M (L^2(M_n)) = N' \cap M_{2n}$$
$$ {}_M B_N (L^2(M_n)) = M' \cap M_{2n+1}$$
$$ {}_M B_M (L^2(M_n)) = M' \cap M_{2n}$$

From Blatt 8, Aufgabe 3 we know: $\exists p_1, \dots p_n \in N' \cap M$ pairwise orthogonal projections $\Rightarrow [M:N] \geq n^2$

Since $[M:N] < \infty$, there are only finitely many orthogonal projections in $N' \cap M$. Thus, all the above relative commutants are finite-dimensional and hence ${}_N L^2(M_n)_M \cong \bigoplus {}_N H_{i\,M}^{\oplus n_i}$, where ${}_N H_{i\,M}$ are irreducible N-M submodules

Put $\mathcal{G}_0 := \{$ all irreducible N-N submodules of ${}_N L^2(M_n)_N \mid n \geq 1\}$
(up to equivalence)

$\mathcal{G}_1 := \{$ all irred. N-M submodules of ${}_N L^2(M_n)_M \mid n \geq 0\}$

Define a graph on vertices $\mathcal{G}_0 \cup \mathcal{G}_1$ with edges as follows:

${}_N H_M \in \mathcal{G}_1 \Rightarrow {}_N H_M$ decomposes into N-N-modules (irred.).

Now, draw n edges between an irreducible N-N-module if it appears n times in ${}_N H_M$.

number of edges between x and y

The bipartite graph $/\mathcal{G}$ yields a matrix $M_{\mathcal{G}} := (m_{xy})_{x \in \mathcal{G}_0,\, y \in \mathcal{G}_1}$

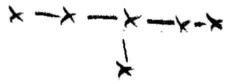
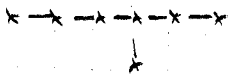
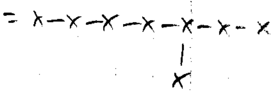
4.1 Theorem: $N \subseteq M$ subfactor, $[M:N]<\infty$, $\mathcal{G}$ the associated graph.

(a) $M_{\mathcal{G}}$ yields $\tilde{M}_{\mathcal{G}} \in B(\ell^2(\mathcal{G}_1), \ell^2(\mathcal{G}_0))$ with $\|\tilde{M}_{\mathcal{G}}\|^2 \le [M:N]$

(b) Bipartite graphs with $\|\tilde{M}_{\mathcal{G}}\| \le 2$ are classified:

$A_n =$  n vertices

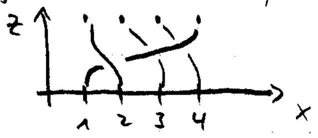
$D_n =$  n vertices

$E_6 =$ , $E_7 =$ , $E_8 =$ 

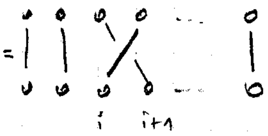
Hence, if $[M:N] \le 4$, then $\mathcal{G}$ must be one of the above graphs (see 2.9).

5 Knots and subfactors

The braid group B_n: An n-braid is a collection of n non-crossing curves $k_i : [0,1] \to \mathbb{R}^3$, $i=1,\dots n$ such that $k_i(0) = (i,0,0)$, $k_i(1) = (\sigma(i),0,1)$ for $\sigma \in S_n$, the z-component of k_i is strictly increasing. Everything is up to isotopy.



Composition of the braids is the group multiplication, the inverse element is given by reflection at the x-axis.

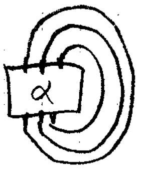
The group B_n is generated by $\sigma_1, \dots \sigma_{n-1}$ with $\sigma_i =$ 

We have $B_n = \langle \sigma_1, \dots \sigma_{n-1} \mid \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \rangle$

5.2 Def: A knot is a homeomorphic image of the circle in $\mathbb{R}^3$.

An n-link is the union of n pairwise disjoint knots.

knot:  2-link: 

The closure of braid $\alpha \in B_n$ is , which gives a link.

5.3 Theorem (Alexander): For every n-link, there is a braid in some B_m such that the closure is this link.

We want to have an invariant for links, i.e. we need a map $\underset{\uparrow \text{link}}{L} \longmapsto P_L$ with $L \underset{\uparrow \text{isotopic}}{\cong} K$, then $P_L = P_K$.

5.4 Def: Two braids $\alpha^{(n)} \in B_n$ and $\beta^{(m)} \in B_m$ are related by a *Markov move* of

(a) type I, if $n = m$ and $\exists g \in B_n$ such that $g \alpha^{(n)} g^{-1} = \beta^{(n)}$

(b) type II, if $\begin{cases} m = n+1 \text{ and } & \beta^{(n+1)} = (\alpha^{(n)} \oplus |)(\sigma_n^{(n)} \oplus |)^{\pm 1} \\ \text{or} & \\ m = n-1 \text{ and } & \beta^{(m)} \oplus | = \alpha^{(n)} (\sigma_{n-1}^{(n-1)} \oplus |)^{\pm 1} \end{cases}$

5.5 Theorem (Markov): Two braids $\alpha^{(n)} \in B_n$ and $\beta^{(m)} \in B_m$ have equivalent link closures if and only if there exists a finite sequence of braids $\alpha^{(n)} = \alpha_0, \alpha_1, \ldots \alpha_k = \beta^{(m)}$ such that α_i and α_{i+1} are related by a *Markov move*.

Let $q \in \{\exp(\pm \frac{2\pi i}{n}) \mid n = 3, 4, \ldots\}$, $\tau = (q + q^{-1} + 2)^{-1}$.
Then there is a subfactor $N \subseteq M$ with $[M:N] = \tau$.
Let $N \subseteq M = M_0 \overset{e_1}{\subseteq} M_1 \overset{e_2}{\subseteq} M_2 \subseteq \ldots$ be its tower and put $g_i := q^{\frac{1}{2}}((q+1)e_i - 1) \in M_n$ for $i = 1, \ldots n$.
Then the g_i are unitaries satisfying the braid relations, i.e. we have a group homomorphism $\pi_n : \begin{array}{l} B_n \longrightarrow M_n \\ \sigma_i \longmapsto g_i \end{array}$.

Put $P_n(\alpha) := (-(q^{\frac{1}{2}} + q^{-\frac{1}{2}}))^{n-1} \operatorname{tr}_{M_n}(\pi_n(\alpha))$.
Check that P_n does not change under Markov moves, i.e. if two links are isotopic, their P_n coincide.
$\Rightarrow$ we have a new invariant for links!