UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 1
for the tutorial on *Tuesday, May 10* (in SR 6)

Exercise 1.

(a) Let φ be a linear functional on some $B(\mathcal{H})$. Prove that the following statements are equivalent:

(i) There are $n \in \mathbb{N}$ and vectors $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n \in \mathcal{H}$, such that

$$\varphi(x) = \sum_{k=1}^{n} \langle x\xi_k, \eta_k \rangle \qquad \text{for all } x \in B(\mathcal{H}).$$

(ii) φ is continuous with respect to the weak operator topology.

(iii) φ is continuous with respect to the strong operator topology.

Show that the equivalence of (ii) and (iii) still holds for a linear functional φ defined on any von Neumann algebra $M \subset B(\mathcal{H})$.

In fact, for any linear functional φ on a von Neumann algebra $M \subset B(\mathcal{H})$, the following statements are equivalent (see Blackadar, Theorem III.2.1.4):

(i) There are sequences $(\xi_k)_{k\in\mathbb{N}}$, $(\eta_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ with $\sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty$ and $\sum_{k=1}^{\infty} \|\eta_k\|^2 < \infty$, such that $\varphi(x) = \sum_{k=1}^{\infty} \langle x\xi_k, \eta_k \rangle$ for all $x \in M$.

(ii) φ, restricted to the unit ball of M, is continuous with respect to the weak operator topology.

(iii) φ, restricted to the unit ball of M, is continuous with respect to the strong operator topology.

(iv) φ is normal.

If φ is a state, these are also equivalent to

(v) There is an orthogonal sequence $(\xi_k)_{k\in\mathbb{N}}$ of vectors in $\mathcal{H}$ with $\sum_{k=1}^{\infty} \|\xi_k\|^2 = 1$, such that $\varphi(x) = \sum_{k=1}^{\infty} \langle x\xi_k, \xi_k \rangle$ for all $x \in M$.

(vi) φ is completely additive, i.e., whenever $(p_i)_{i\in I}$ is a family of mutually orthogonal projections in M, then $\varphi\big(\sum_{i\in I} p_i\big) = \sum_{i\in I} \varphi(p_i)$.

(b) Let M be a finite factor and let τ be the unique norm-continuous trace on M (see Theorem 1.11). As usually, we denote by $L^2(M) = L^2(M, \tau)$ the complex Hilbert space obtained by completion of M with respect to the inner product $\langle \cdot, \cdot \rangle$ induced by τ, i.e. $\langle x, y \rangle := \tau(xy^*)$ for all $x, y \in M$. The corresponding norm on $L^2(M)$ will be denoted by $\| \cdot \|_2$.

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Consider now the unit ball $B := \{x \in M \mid \|x\| \leq 1\}$ with respect to the operator norm $\|\cdot\|$ on M. Show that B, endowed with the metric induced by $\|\cdot\|_2$, is a complete metric space and that the topology on B induced by $\|\cdot\|_2$ is the same as the strong operator topology.

Exercise 2. Let $M \subset B(\mathcal{H})$ be a type II_1-factor with trace τ, acting on some Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where M possesses a cyclic and separating vector Ω such that $\tau(x) = \langle x\Omega, \Omega \rangle$ for all $x \in M$. Denote by M' the commutant of M and let $J : \mathcal{H} \to \mathcal{H}$ be the *antilinear unitary involution* determined by $J(x\Omega) = x^*\Omega$ for all $x \in M$. Prove the following statements:

(a) For all $\xi, \eta \in \mathcal{H}$, we have $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$.

(b) For all $x, a \in M$, it holds true that $JxJ(a\Omega) = ax^*\Omega$.

(c) If $x \in M'$ is given, we have $Jx\Omega = x^*\Omega$.

Deduce finally that $JMJ = M'$.

Hint: Switch the roles of M and M'. What does (c) tell us about this case?

Exercise 3. Fix any integer $m \in \mathbb{N}$, $m \geq 2$. Consider the chain of inclusions

$$M_m(\mathbb{C}) \hookrightarrow M_{m^2}(\mathbb{C}) \hookrightarrow M_{m^3}(\mathbb{C}) \hookrightarrow \ldots \hookrightarrow M_{m^n}(\mathbb{C}) \hookrightarrow M_{m^{n+1}}(\mathbb{C}) \hookrightarrow \ldots,$$

which are given by

$$M_{m^n}(\mathbb{C}) \hookrightarrow M_{m^{n+1}}(\mathbb{C}), \quad B \mapsto \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}.$$

(a) Justify that its union $M^{(m)} := \bigcup_{n\in\mathbb{N}} M_{m^n}(\mathbb{C})$ is a complex unital algebra and show that there exists a (well-defined!) tracial linear functional $\tau^{(m)} : M^{(m)} \to \mathbb{C}$ such that $\tau^{(m)}(B) = \mathrm{tr}_{m^n}(B)$ holds for any $B \in M_{m^n}(\mathbb{C})$. Recall that tr_{m^n} denotes the normalized trace on $M_{m^n}(\mathbb{C})$.

(b) Denote by $\mathcal{H}^{(m)}$ the Hilbert space which is obtained by completion of $M^{(m)}$ with respect to the inner product given by $\langle A, B \rangle_{(m)} = \tau^{(m)}(AB^*)$. Prove that each $B \in M^{(m)}$ induces a bounded linear operator on $\mathcal{H}^{(m)}$, i.e., we can view $M^{(m)} \subset B(\mathcal{H}^{(m)})$.

(c) Consider the von Neumann algebra $\mathcal{R} := \overline{M^{(m)}}^{\mathrm{sot}} \subset B(\mathcal{H}^{(m)})$. Show that there exists a unique normal tracial state τ on $\mathcal{R}$.

(d) Prove that $\mathcal{R}$ is a type II_1-factor.

Hint: Since the center $Z(\mathcal{R}) := \mathcal{R} \cap \mathcal{R}'$ of $\mathcal{R}$ is generated by its positive elements, factoriality follows as soon as we have shown that any positive $z \in Z(\mathcal{R})$ is a positive multiple of 1. For doing so, use the result obtained in (c).

It is a non-trivial result of Murray and von Neumann that $\mathcal{R}$ does not depend on the special choice of m. In fact, the obtained von Neumann algebra $\mathcal{R}$ is the hyperfinite II_1-factor. To see that $\mathcal{R}$ is isomorphic to $L(S_\infty)$, as the hyperfinite II_1-factor was introduced in the lecture, is again a non-trivial result of Murray and von Neumann.