UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 2
for the tutorial on *Tuesday, May 17* (in SR 6)

Exercise 1. Let $M \subset B(\mathcal{H})$ be von Neumann algebra on some Hilbert space $\mathcal{H}$ and let $p \in M$ be a non-zero projection. Prove the following statements:

(a) We have $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on the Hilbert space $p\mathcal{H} = \operatorname{ran}(p)$. Thus pMp and $M'p$ are both von Neumann algebras on $p\mathcal{H}$.

Hint: First show that $(pM')' = pMp$ holds. Conclude by proving that any unitary $u \in (pMp)'$ can be extended to an isometry $\tilde{u} : \mathcal{K} \to \mathcal{K}$ on the Hilbert space $\mathcal{K} := \overline{Mp\mathcal{H}} \subset \mathcal{H}$, such that $\tilde{u}q \in M'$ holds for q being the orthogonal projection from $\mathcal{H}$ onto $\mathcal{K}$. For this purpose, check that $q \in Z(M)$.

(b) If M is a factor, then pMp and pM' are both factors on $p\mathcal{H}$. Moreover, the map

$$\Phi : \ M' \to M'p, \ x \mapsto xp$$

is a weakly continuous $*$-algebra isomorphism.

Hint: Use the general fact (which was proven in (a)) that the orthogonal projection q from $\mathcal{H}$ onto $\mathcal{K} = \overline{Mp\mathcal{H}}$ belongs to $Z(M)$.

(c) If M is a factor and if $x \in M$ and $y \in M'$ are given, then $xy = 0$ implies that $x = 0$ or $y = 0$.

(d) If M is a factor, then $M \cup M'$ generates $B(\mathcal{H})$ as a von Neumann algebra.

(e) If M is a type II_1-factor, then $pMp \subset B(p\mathcal{H})$ is also a type II_1-factor.

Exercise 2. Let M be a type II_1-factor and denote by τ_M its canonical trace. Prove the following properties of the coupling constant:

(a) If $(\mathcal{H}_i)_{i\in I}$ is a family of M-modules over a countable index set I, we have that

$$\dim_M\Big(\bigoplus_{i\in I} \mathcal{H}_i\Big) = \sum_{i\in I} \dim_M(\mathcal{H}_i).$$

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(b) If $\mathcal{H}$ is an M-module and $p \in M$ a projection, then it holds true that

$$\dim_{pMp}(p\mathcal{H}) = \frac{1}{\tau_M(p)} \dim_M(\mathcal{H}).$$

(c) Consider the commutant M' of M with respect to its standard representation on $L^2(M)$. If $q \in M'$ is a projection, we have that

$$\dim_M(qL^2(M)) = \tau_{M'}(q).$$

(d) Assume that $\mathcal{H}$ is an M-module for which M' is also a type II_1-factor. We denote the canonical trace of M' by $\tau_{M'}$. For any $p \in M'$, it holds true that

$$\dim_{Mp}(p\mathcal{H}) = \tau_{M'}(p) \dim_M(\mathcal{H}).$$

Exercise 3. Consider the type I_n-factor $M = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and denote by tr_n its normalized trace.

(a) Discuss the statements (a) – (d) of Exercise 1 in each of the two cases

$$\mathcal{H} = \mathbb{C}^n \qquad \text{and} \qquad \mathcal{H} = L^2(M)$$

(b) It is known that each representation of M on a finite dimensional Hilbert space $\mathcal{H}$ is unitarily equivalent (in analogy to Definition 2.8) to a representation of the form

$$M \to B(\mathbb{C}^n \otimes \mathbb{C}^k) = M_n(\mathbb{C}) \otimes M_k(\mathbb{C}),\ x \mapsto x \otimes 1$$

for some $k \in \mathbb{N}_0$. In this case, we put

$$\dim_M(\mathcal{H}) := \frac{k}{n}.$$

What are the correct analogues of the properties (a) – (d) in Exercise 2?