UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 3
for the tutorial on *Tuesday, June 7* (in SR 6)

Exercise 1. Let H and G be discrete i.c.c. groups, such that H is a subgroup of G. We denote by $[G : H]$ the group theoretic index of H in G, i.e. the number of (left or right) cosets of H in G. Recall that left and right cosets of H in G are of the form $gH = \{gh|\ h \in H\}$ and $Hg = \{hg|\ h \in H\}$ for $g \in G$, respectively, and that their number is always the same.

(a) Justify that $\ell^2(G)$ provides an $L(H)$-module and prove that its $L(H)$-dimension is given by
$$\dim_{L(H)}(\ell^2(G)) = [G : H].$$

(b) Consider the group factor $L(G)$ and denote by τ its canonical trace. Show that
$$L^2(L(G), \tau) \qquad \text{and} \qquad \ell^2(G)$$
are isomorphic as $L(G)$-modules.

(c) Show that $L(H)$ can be considered as a subfactor of $L(G)$ and deduce for the corresponding Jones index that
$$[L(G) : L(H)] = [G : H].$$

Exercise 2. Let $M \subseteq B(\mathcal{H})$ be a finite dimensional von Neumann algebra.

(a) Prove that pMp is a factor on $p\mathcal{H}$ for each minimal projection p in the center $Z(M)$.

(b) Show that the center $Z(M)$ is a finite dimensional abelian von Neumann algebra, which can be written as
$$Z(M) = \bigoplus_{i=1}^{k} (\mathbb{C}p_i),$$
where $p_1, \dots, p_k$ denote the minimal projections in $Z(M)$.

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(c) Deduce that there are $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$, such that M is isomorphic to

$$\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}).$$

In fact, there are Hilbert spaces $\mathcal{K}_1, \ldots, \mathcal{K}_k$ and a unitary $u : \bigoplus_{i=1}^{k}(\mathbb{C}^{n_i} \otimes \mathcal{K}_i) \to \mathcal{H}$, such that

$$u^* M u = \bigoplus_{i=1}^{k}(B(\mathbb{C}^{n_i}) \otimes 1).$$

Hint: You may use without giving a proof that for any type I_n-factor M on a Hilbert space $\mathcal{H}$, there exists a Hilbert space $\mathcal{K}$ and a unitary $u : \mathbb{C}^n \otimes \mathcal{K} \to \mathcal{H}$, such that $uMu^* = B(\mathbb{C}^n) \otimes 1$.

Exercise 3.

(a) Let M be a factor of type I_n. Prove that any subfactor N of M is of type I_m for some integer m dividing n. Moreover, show that all subfactors N of M of type I_m are uniquely determined, up to conjugation by unitaries in M, by the integer $k > 0$ such that pMp is a factor of type I_k for some minimal projection $p \in N$ and $mk = n$.

(b) Let $N \subseteq M$ be finite dimensional von Neumann algebras. Let $p_1, \ldots, p_m$ be the minimal central projections of M and $q_1, \ldots, q_n$ those of N. For each $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, $p_j q_i M q_i p_j$ yields a factor with subfactor $p_j q_i N$, to which we may associate an integer $k_{i,j}$ according to (a). We form the matrix

$$\Lambda = (k_{i,j})_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}.$$

Compute Λ for $M = M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ and the subalgebra N of matrices of the form

$$\begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z \end{pmatrix} \oplus \begin{pmatrix} X & 0 \\ 0 & z \end{pmatrix} \qquad \text{with } z \in \mathbb{C} \text{ and } X \in M_2(\mathbb{C}).$$

Often, the matrix

$$\Lambda = (k_{i,j})_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$$

is represented by a bipartite graph $G = (V, E)$ on the vertex set $V = P \dot{\cup} Q$ with $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$, which has $k_{i,j}$ edges between q_i and p_j. The obtained graph G is called the *Bratteli diagram* for $N \subseteq M$.

(c) Show that $k_{i,j} = \mathrm{Tr}(p_j e_i)$ holds, if e_i is a minimal projection in the factor $q_i N$. Note that Tr denotes here the unnormalized trace on $p_j M p_j$, which is isomorphic $M_{m_j}(\mathbb{C})$ for some $m_j \in \mathbb{N}$.