UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 4
for the tutorial on *Tuesday, June 21* (in SR 6)

Exercise 1. Let $p, q \in B(\mathcal{H})$ be orthogonal projections on a separable complex Hilbert space $\mathcal{H}$.

(a) Show that

$$\operatorname*{s\text{–}lim}_{n\to\infty} (pqp)^n = p \wedge q.$$

Hint: First show (for instance by using functional calculus) that the sequence $((pqp)^n)_{n=1}^\infty$ converges strongly to some projection $e \in B(\mathcal{H})$. Finally, in order to identify e as $p \wedge q$, consider expressions of the form $(pqp)^m q (pqp)^n$ for $m, n \in \mathbb{N}$.

(b) Deduce the formula stated in Remark 3.10 (3), i.e., prove that

$$\operatorname*{s\text{–}lim}_{n\to\infty} (pq)^n = p \wedge q.$$

(c) Discuss the statements (a) and (b) in the case $\mathcal{H} = \mathbb{C}^3$ for the projections

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q = u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} u^*, \quad \text{where} \quad u = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

for some $0 < \theta < \pi$.

Exercise 2. Let $(S_n)_{n=0}^\infty$ be the sequence of *Chebyshev polynomials of the second kind*, which are recursively defined by $S_0(x) = 1$, $S_1(x) = x$ and

$$x S_n(x) = S_{n+1}(x) + S_{n-1}(x) \qquad \text{for all } n \geq 1.$$

Prove the following statements:

(a) For all $n \geq 0$ and all $0 < \theta < \pi$, it holds true that

$$S_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

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(b) We have for all $n, m \geq 0$ that

$$\int_{-2}^{2} S_n(x) S_m(x) \frac{1}{2\pi} \sqrt{4 - x^2} \, dx = \delta_{n,m}.$$

(c) For all $x \in [-2, 2]$ and all $z \in \mathbb{C}$ with $|z| < 1$, we have that

$$\frac{1}{1 - xz + z^2} = \sum_{n=0}^{\infty} S_n(x) z^n.$$

(d) For $x, y \in [-2, 2]$ and all $n \geq 0$, it holds true that

$$\frac{S_n(x) - S_n(y)}{x - y} = \sum_{k=1}^{n} S_{k-1}(x) S_{n-k}(y).$$