UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
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Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 5
for the tutorial on *Tuesday, July 12* (in SR 6)

Out of the following six exercises, choose these two that you like at most. This means that we do not expect that you work out a solution for all problems!

Given any finite dimensional von Neumann algebra M, we know from Exercise 2 (c), Assignment 3, that

$$M \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C})$$

for some $l \in \mathbb{N}$ and $m_1, \dots, m_l \in \mathbb{N}$. For any choice of $\vec{t} = (t_1, \dots, t_l)^T \in \mathbb{R}_+^l$, where $\mathbb{R}_+ := (0, \infty)$, we can thus introduce a faithful trace τ on M by

$$\tau := (t_1 \operatorname{Tr}_{m_1}) \oplus \cdots \oplus (t_l \operatorname{Tr}_{m_l}).$$

In fact, it is easy to see that any faithful trace τ on M arises in this way, and in this case the corresponding vector $\vec{t}$ is called the *trace vector of* τ. Obviously, the trace τ is normalized (i.e. $\tau(1) = 1$), if and only if $t_1 m_1 + \cdots + t_l m_l = 1$ holds.

Exercise 1.

(a) In Exercise 3 (b), Assignment 3, we have constructed for any inclusion $N \subseteq M$ of finite dimensional von Neumann algebras a matrix Λ_N^M. Consider now finite dimensional von Neumann algebras $N \subseteq M \subseteq P$. Prove that the matrices corresponding to these inclusions satisfy the relation

$$\Lambda_N^P = \Lambda_N^M \Lambda_M^P.$$

(b) Take finite dimensional von Neumann algebras $N \subseteq M$, satisfying

$$N \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \qquad \text{and} \qquad M \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C}),$$

with the matrix

$$\Lambda_N^M = (\Lambda_{ij})_{\substack{i=1,\dots,k\\ j=1,\dots,l}}$$

constructed according to Exercise 3 (b), Assignment 3. Moreover, let τ_N and τ_M be a faithful tracial states on N and M, respectively, with corresponding trace vectors $\vec{s} = (s_1, \dots, s_k)^T$ and $\vec{t} = (t_1, \dots, t_l)^T$. Prove that $\tau_M|_N = \tau_N$ if and only if $\Lambda_N^M \vec{t} = \vec{s}$.

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It can be shown that the basic construction works equally well in the non-factor case. More precisely, in the situation of the previous exercise and under the assumption that $\tau_M|_N = \tau_N$ holds, we can find a projection $e_N \in B(L^2(M, \tau_M))$, such that $e_N(x\Omega) = E_N(x)\Omega$ holds for all $x \in M$, where $\Omega = \hat{1} \in L^2(M, \tau_M)$ and E_N denotes the conditional expectation from M to N as in Theorem 3.2. We consider then the von Neumann algebra $\langle M, e_N \rangle \subseteq B(L^2(M, \tau_M))$ generated by M and e_N.

Lemma (Jones, 1983)**.** *Let $p_1, \ldots, p_k$ be the minimal central projections of N. Then*

(i) $Jp_1J, \ldots, Jp_kJ$ *are the minimal central projections of* $\langle M, e_N \rangle$,

(ii) $\Lambda_M^{\langle M, e_N \rangle} = (\Lambda_N^M)^T$ *(with the obvious identification of the indices* $p_i \leftrightarrow Jp_iJ$*)*,

(iii) $e_N Jp_iJ = e_N p_i$,

(iv) $x \mapsto e_N x Jp_iJ$ *is an isomorphism from* p_iN *onto* $(e_N Jp_iJ)\langle M, e_N \rangle (e_N Jp_iJ)$.

Exercise 2. Consider the finite dimensional von Neumann algebras $N \subseteq M$ given by

$$\begin{aligned} \mathbb{C} \oplus \mathbb{C} &\xrightarrow{\cong} N \quad \subseteq \quad M := M_2(\mathbb{C}) \\ z_1 \oplus z_2 &\mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \end{aligned}$$

We endow M with the usual trace $\tau_M = \mathrm{tr}_2$ and N with the restriction $\tau_N = \tau_M|_N$. Compute $e_N \in B(L^2(M, \tau_M))$ and check explicitly the validity of the statements made in the lemma above.

Following Jones, we call a faithful tracial state τ on $M_1 = \langle M, e_N \rangle$ a *(λ, P)-trace*, for $\lambda > 0$ and a subalgebra P of M_1, if τ extends τ_M and $\tau(e_N x) = \lambda \tau_M(x)$ holds for all $x \in P$.

Theorem (Jones, 1983)**.** *Given $\lambda > 0$, there exists a (λ, M)-trace on M_1, if and only if*

$$\Lambda^T \Lambda \vec{t} = \lambda^{-1} \vec{t} \qquad \text{and} \qquad \Lambda \Lambda^T \vec{s} = \lambda^{-1} \vec{s}, \qquad \text{where } \Lambda = \Lambda_N^M. \tag{1}$$

Exercise 3.

(a) Show that a (λ, N)-trace on M_1 is also a (λ, M)-trace on M_1.

(b) Prove the above theorem of Jones.

(c) Show that if condition (1) is satisfied for finite dimensional von Neumann algebras $N \subseteq M$, endowed with traces such that $\tau_M|_N = \tau_N$ holds, then the basic construction can be iterated in the sense that there is a (λ, M)-trace on $M_1 = \langle M, e_N \rangle$, a (λ, M_1)-trace on $M_2 = \langle M_1, e_M \rangle$, and so on.

It was observed by Jones that the projections appearing in the Jones tower

$$N \subseteq M \overset{e_1 = e_N}{\subseteq} M_1 \overset{e_2 = e_M}{\subseteq} M_2 \overset{e_3 = e_{M_1}}{\subseteq} M_3 \overset{e_4 = e_{M_2}}{\subseteq} \ldots$$

constructed according to part (c) of the previous exercise can be used to build a subfactor $P_\lambda \subseteq P$ with Jones index $[P : P_\lambda] = \lambda^{-1}$. In fact, it can be shown that P is isomorphic to the hyperfinite II_1-factor.

Exercise 4. Let $n \in \mathbb{N}$ with $n \geq 2$ be fixed. Consider the symmetric matrix $\Lambda = (\Lambda_{ij})_{i,j=1}^n$ defined by

$$\Lambda_{ij} := \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{else} \end{cases} \qquad \text{for } i,j = 1,\dots,n,$$

i.e., we have

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

(a) Prove that the eigenvalues of Λ are precisely the zeros of the n-th Chebyshev polynomial S_n of the second kind (cf. Exercise 2, Assignment 4), i.e.

$$\Big\{2\cos\Big(\frac{k\pi}{n+1}\Big) \,\Big|\, k = 1,\dots,n\Big\},$$

where an eigenvector corresponding to the eigenvalue $\lambda_k = 2\cos\big(\frac{k\pi}{n+1}\big)$ is given by

$$\vec{t}_k = \Big(\sin\Big(\frac{k\pi}{n+1}\Big), \sin\Big(\frac{2k\pi}{n+1}\Big), \dots, \sin\Big(\frac{nk\pi}{n+1}\Big)\Big)^T.$$

(b) Deduce that all values in

$$\Big\{4\cos^2\Big(\frac{\pi}{n+1}\Big) \,\Big|\, n \geq 2\Big\}$$

show up as the Jones index for some subfactor of the hyperfinite II_1-factor.

It is worth to point out that (1) gives an interesting connection to the famous Perron-Frobenius Theorem. More precisely, the existence of a positive eigenvector $\vec{t}$ for the matrix $P = \Lambda\Lambda^T$ (or analogously $\vec{s}$ for $P = \Lambda^T\Lambda$) implies that the corresponding eigenvalue λ^{-1} determines its norm by $\|\Lambda\|^2 = \|P\| = \lambda^{-1}$ and hence the Jones index of the constructed subfactor $P_\lambda \subseteq P$, i.e. $\|\Lambda\|^2 = [P : P_\lambda]$. However, for this purpose, we do not need the Perron-Frobenius Theorem in full generality. Hence, a more specialized proof (which nevertheless follows ideas of the general proof) is more appropriate.

Exercise 5. Let a real matrix $P = (P_{ij})_{i,j=1}^n \in M_n(\mathbb{R})$ be given, which is both symmetric (i.e. $P^T = P$) and non-negative (i.e. $P_{ij} \geq 0$ for all $i,j = 1,\dots,n$). Moreover, assume that there exists a real eigenvector $y = (y_1,\dots,y_n)^T$ of P, which satisfies $y_1,\dots,y_n > 0$, with corresponding eigenvalue $\lambda \geq 0$.

(a) On the set

$$\Gamma_n := \{x = (x_1,\dots,x_n) \in \mathbb{R}^n \mid x_1,\dots,x_n > 0\}$$

consider the function

$$L:\ \Gamma_n \to [0,\infty),\ x \mapsto \max\{s \geq 0 \mid sx \leq Px\},$$

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where $x \leq x'$ for real vectors $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$ means that $x_i \leq x'_i$ holds for all $i = 1, \dots, n$. Prove that

$$\sup_{x \in \Gamma_n} L(x) = \lambda = L(y).$$

Hint: Consider the inner product of $\langle x, y \rangle$ for $x \in \Gamma_n$ and check that we always have $\langle x, y \rangle > 0$ in this case.

(b) Deduce that $\|P\| = \lambda$.

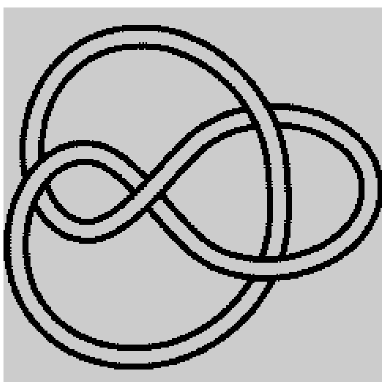
Hint: Note that if $\lambda_1, \dots, \lambda_n$ are the ordered eigenvalues of any symmetric real matrix P, listed according to their multiplicity, then $\|P\| = \max\{|\lambda_1|, \dots, |\lambda_n|\}$.

Exercise 6.

(a) Find a braid b whose closure $\hat{b}$ yields the following link and compute its Jones polynomial $V_{\hat{b}}(t)$.

Hint: Note that there are actually *two different* Jones polynomials related to the picture above, depending on the choice of an orientation on both of its components, since this will change the corresponding element in the braid group.

(b) Find a braid b whose closure $\hat{b}$ yields the following knot and compute its Jones polynomial $V_{\hat{b}}(t)$.



Hint: Choose any point P in the plane, which does not belong to the given projection of the knot, and fix an orientation of the knot. Try to deform the knot until its orientation on any subarc goes in mathematical positive sense around P. Decompose the obtained projection of the knot in sectors around P, such that each sector contains at most one crossing of the knot.