UNIVERSITÄT DES SAARLANDES
FACHRICHTUNG 6.1 – MATHEMATIK
Prof. Dr. Roland Speicher
M.Sc. Tobias Mai

Assignments for the lecture on
von Neumann algebras, subfactors, and planar algebras
Summer term 2016

Assignment 6
for the tutorial on ***Thursday, July 28*** (in SR 6)

Exercise 1 (*von Neumann mean ergodic theorem*)**.** Let $\mathcal{H}$ be a separable complex Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n \xi = \pi(\xi) \tag{1}$$

holds for any $\xi \in \mathcal{H}$, where π denotes the orthogonal projection from $\mathcal{H}$ onto the closed subspace $\mathcal{H}^U$ of all U-invariant vectors in $\mathcal{H}$, i.e. $\mathcal{H}^U := \{\xi \in \mathcal{H} |\ U\xi = \xi\}$.

Hint: Consider the (possibly non-closed) subspace $\mathcal{W} := \{U\xi - \xi |\ \xi \in \mathcal{H}\}$ of $\mathcal{H}$ and show that

(a) $\mathcal{W}$ and $\mathcal{H}^U$ are orthogonal,

(b) formula (1) holds separately on $\mathcal{H}^U$ and $\mathcal{W}$ (and hence also on $\mathcal{H}^U + \mathcal{W}$),

(c) formula (1) holds on $\overline{\mathcal{H}^U + \mathcal{W}}$,

(d) we have $\overline{\mathcal{H}^U + \mathcal{W}} = \mathcal{H}$.

Exercise 2. By Lemma 7.1 and Lemma 7.2 of the lecture we know that any (faithful) trace tr on the Temperley-Lieb algebra $\mathrm{TL}(\lambda) = \bigcup_{n\geq 0} \mathrm{TL}_n(\lambda)$

... is fully determined by its action on cyclically reduced words

$$w = e_{i(1)} e_{i(2)} \cdots e_{i(m)} \qquad \text{with } |i(j) - i(k)| \geq 2 \text{ whenever } j \neq k,$$

... and must take the same value on all cyclically reduced words of the same length m.

Use the von Neumann mean ergodic theorem stated in Exercise 1 in order to show that tr attains the value λ^m on each cyclically reduced word of length m (and hence coincides with the canonical trace τ on $\mathrm{TL}(\lambda)$).

Hint: Generalize the example discussed in the proof of Proposition 7.3: given any cyclically reduced word $w = e_{i(1)} e_{i(2)} \cdots e_{i(m)}$, choose some appropriate index set $I \subseteq \mathbb{N}$ and consider the unital complex algebra $A_I \subseteq \mathrm{TL}(\lambda)$ generated by $\{e_i |\ i \in I\}$. Find an interesting unitary operator on the corresponding Hilbert space $\mathcal{H} = L^2(A_I, \mathrm{tr}|_{A_I})$.