Saarland University

Lecture Notes on

Von Neumann Algebras, Subfactors, Knots and Braids, and Planar Algebras

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Abstract: These are the lecture notes of a course which was given in the summer term 2016 at Saarland University by Roland Speicher. It covers the basic work of Vaughan Jones on subfactors, Jones index, the relation to knots and the Jones polynomial, and planar algebras. It assumes some familiarity with basic functional analysis and operator algebraic notions; the starting point are von Neumann algebras – for those the basic definitions and facts are stated, though in general without proofs. After this, for the representation theory of von Neumann algebras and subfactors etc. the main statements are usually given with proof. The exercises in Section 12 were prepared by Tobias Mai. The lectures and the present manuscript benefitted a lot from the fundamental papers, various survey articles and lecture manuscripts by Jones. For the first parts, the unpublished lecture notes (written by Dietmar Bisch) of the course of Sorin Popa "Lectures on von Neumann algebras: type II_1 factors" from Fall 1987 and Winter 1988 at UCLA were also very helpful.

Contents

1 Motivation and Survey

Let us start with giving a survey what we will cover. Precise definitions will be given later.

We will be interested in von Neumann algebras $M \subset \mathcal{B}(\mathcal{H})$. $\mathcal{H}$ is here a Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the bounded linear operators on $\mathcal{H}$.

Operator algebras are $*$- algebras of bounded operators on a Hilbert space which are closed in some canonical topologies. The two main classes of operator algebras are C^*-algebras, which are closed in the operator norm, and von Neumann algebras, which are closed in the weak operator topology; the first topology is the operator version of uniform convergence, the latter of pointwise convergence. We will not address C^*-algebras here, but concentrate on von Neumann algebras. Von Neumann algebras are quite large objects and their classification is notoriously difficult.

With M' we denote the commutant (on $\mathcal{H}$) of our von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$. Basic building blocks for von Neumann algebras are factors, i.e., M with the property that $M \cap M' = \mathbb{C}1$. The early classification of von Neumann algebras by Murray and von Neumann divided the world of factors into types I, II, and III. The type I factors are the trivial ones, namely $M = \mathcal{B}(\mathcal{H})$. Type III are too "exotic" for us; we will only be interested in type II. Actually, type II splits into two subcases, II_1 and II_∞, and we will only consider II_1. They are characterized by being infinite-dimensional and having a (unique) trace $\text{tr} : M \to \mathbb{C}$. Those are the really cool factors.

Given such a factor M the first canonical thing to do is to think about representations (as operators on Hilbert spaces) of M. There is actually a canonical "standard" representation, which is given by the GNS (i.e., Gelfand-Naimark-Segal) construction with respect to the trace; there the algebra is acting on itself by multiplication, where the trace is used to define an inner product on the algebra. Note that this representation is bigger than one might expect from matrices. Namely, consider the type I factor $M = M_n(\mathbb{C})$, given by $n \times n$ matrices. The defining representation for this is $M = \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ being n-dimensional; i.e., the representation of having the $n \times n$ matrices act on $\mathbb{C}^n$. However, from our perspective a "better" representation space is $L^2(M, \text{tr}) = \mathbb{C}^{n^2} = \mathbb{C}^n \otimes \mathbb{C}^n$, with

$$\mathcal{B}(L^2(M, \text{tr})) = M_{n^2}(\mathbb{C}) = \underbrace{M_n(\mathbb{C})}_{M} \otimes \underbrace{M_n(\mathbb{C})}_{M'}.$$

In this representation M and its commutant M' are of the same size, and there exists a vector ξ which is both cyclic and separating for M. This is true for the standard representation of any II_1 factor.

There are other represention of M than the standard one on $L^2(M)$, but those are actually quite easy to characterize. A representation is captured by the notion of an M-module which is a Hilbert space $\mathcal{H}$ together with an action of M on $\mathcal{H}$ via a unital $*$-homomorphism $M \to \mathcal{B}(\mathcal{H})$ with some continuity property. For such an M-module one can define the "coupling constant" or the M-dimension of $\mathcal{H}$ as a number $\dim_M \mathcal{H} \in [0, \infty]$. One has $\dim_M L^2(M) = 1$ and this dimension characterizes the representation theory of M: $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$ is equivalent to the fact that the representation of M on $\mathcal{H}$ is unitarily equivalent to the representation of M on $\mathcal{K}$. Hence the representation theory of II_1 factors is kind of trivial. Representations of M are characterized by a number in $[0,\infty]$ (which can be thought of as the relative size of the representation space compared to the standard space $L^2(M)$). The fact that the dimensions characterizing our M-modules can take on all real non-negative values is one of the nice and interesting features of the II_1 world. In the type I analogue the only representations for $M_n(\mathbb{C})$ are of the form $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ and the possible values for $\dim_{M_n(\mathbb{C})} \mathcal{H}$ are given by the discrete multiplicity k of the representation. In contrast, II_1 factors can be thought of as describing a continuous geometry with all possible real dimensions in $[0,1]$.

Let us now consider two factors sitting inside each other and try to characterize their relative position. So we are interested in $N \subseteq M$, where both N and M are II_1 factors with the same identity. We call N then a *subfactor* of M.

For such a situation Jones defined the *index* of N in M by

$$[M:N] = \dim_N L^2(M).$$

This is intended to be a measure of the relative size of N in M. For example, it is quite easy to see that $[M:N]=1$ if and only if $N=M$.

Another example of a subfactor where we can directly determine the index is the following. Let N be a type II_1 factor and define $M = M_k(N) \equiv N \otimes M_k(\mathbb{C})$. Then we have an embedding

$$N \to M, \quad x \mapsto x \otimes 1$$

with the identification

$$x \otimes 1 \equiv \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & x & 0 \\ 0 & \cdots & 0 & x \end{bmatrix}.$$

Then we have

$$L^2(M) = L^2(N \otimes M_k(\mathbb{C})) = \bigoplus_{j=1}^{k^2} L^2(N)$$

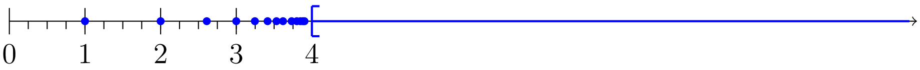
and hence $[M:N] = k^2$.

It is not obvious how to generate other examples of indices. By the continuous nature of II_1 factors one might expect to find also non-real values of the indices. However, the mixture between a continous and discrete part for all possible values for the index in the following famous result of Vaughan Jones came as a big surprise.

Theorem (Jones 1983)**.**
Let $N \subset M$ be a II_1 subfactor, then the possible values of $[M:N]$ are given by

$$\{4\cos^2(\frac{\pi}{n}) \colon n = 3,4\ldots\} \cup [4,\infty).$$

We can visualize this by the following number line.

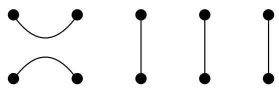


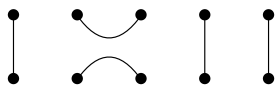
The proof of this theorem above is done via the so-called basic construction, which gives a sequence of projections $e_1, e_2, \ldots$ which satisfy $e_i e_{i\pm1} e_i = \lambda e_i$ and $e_i e_j = e_j e_i$, for $|i-j| \geq 2$, where $\lambda = [M : N]^{-1}$. Those relations are closely related to relations which describe knots and braids. This connection led in the end to the famous *Jones polynomial*, which is a new invariant for knots. The Jones polynomial allows for example to distinguish easily between the left and right trefoil knot by

$$V_L(q) = q^{-1} + q^{-3} - q^{-4}, \quad V_R(q) = q + q^3 - q^4.$$

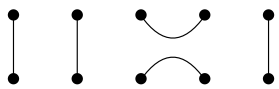
(a) left Trefoil (b) right Trefoil

The connection between the projections e_i on one side and braids and knots on the other side is not coming as a total surprise if one realizes that the relations for the e_i can, up to some rescaling, be visualized in a nice pictorial way. For example, identify E_1, E_2, E_3 in the following way with diagrams.

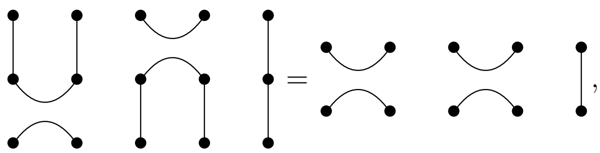
$E_1 =$ 

$E_2 =$ 

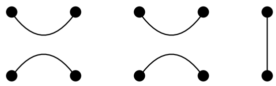
and

$E_3 =$ 

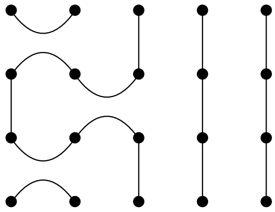
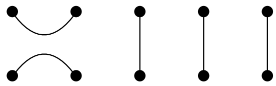
We realize then the multiplication in this representation via a vertical stacking of the graphs with subsequent removal of the middle line of dots. So we get then for example

$E_3 E_1 =$ 

similarly one gets

$$E_1E_3 =$$  .

This shows that we have indeed the relation $E_3E_1 = E_1E_3$ for those diagrams. Furthermore we have

$$E_1E_2E_1 =$$  $=$  $= E_1,$

which gives, after some rescaling, the relation $e_1e_2e_1 = \lambda e_1$.

The algebra generated by such pictures is actually a well know object, the so-called *Temperley-Lieb algebra*. Hence every subfactor $N \subset M$ yields the Temperley-Lieb algebra. This raises the question whether we have more specific diagrammatic descriptions of the fine structure of subfactors. The index $[M : N]$, which corresponds to the Temperley-Lieb algebra, is only a very rough index for subfactors; for the classification of subfactors we want finer information. Motivated by the above diagrammatic representation of the Temperley-Lieb algebra, Jones introduced around 1999 the notion of *planar algebras*. This is on one side intended to provide a diagrammatic way of producing and understanding invariants for subfactors, but allows on the other hand also a rigorous justification for "pictorial" proofs of subfactor theorems.

In this class we will first recall, in Chapter 2, the basic facts about von Neumann algebras and II_1 factors. Then, in Chapter 3, we give the details of the representation theory of II_1 factors. In Chapter 4, we introduce one of our main objects of interests, the Jones index, and derive some of its elementary properties. Chapter 5 deals with the more advanced tools for the investigation of the index, in particular, the basic construction of Jones. Iteration of the basic construction yields then, in Chapter 6, the Jones tower and finally the proof of Jones theorem on the possible values of the index. Chapter 7 addresses the question whether the possible values less than 4 can actually occur and we will investigate the Temperley-Lieb algebra to give the main ideas for a proof of that fact. Whereas up to this point the whole theory was quite analytic, the Temperley-Lieb algebra introduces diagrammatics into the game and we will start to appreciate the idea that pictures might sometimes tell more than abstract formulas. This makes then also the connection to braids and knots. In Chapter 8, we will hence take a break from subfactors and follow up more on the braid connection. This will culminate in the famous Jones polynomial. Though this can in the end be defined without any reference to von Neumann algebras, the latter were essential for coming up with the idea of making this definition. This astonishing relation between two very different subjects - von Neumann algebras and knots - earned Jones the Fields Medal in 1990. In Chapter 9 we will switch back again to our quest of understanding subfactors, but now biased by the idea that the analytic structure might be best encoded in diagrammatic terms. We will follow here

Jones again who introduced the notion of “planar algebras” as the right diagrammatic tool for dealing with invariants of subfactors. Chapters 10 and 11 have more on this. Whereas the early theory about the Jones index and the Jones polynomial is classic and more or less in final form, planar algebras are still a very active subject, where the final word has not yet been spoken.

I thank Vaughan Jones and, in particular, Dietmar Bisch for many discussions on planar algebras and very constructive feedback for the present notes. Much of what I know about planar algebras is due to Dietmar.

2 Von Neumann Algebras and II_1 Factors

In this section we will recall the basic definitions and facts about von Neumann algebras and II_1 factors; mostly without proofs.

Definition 2.1.

1) A *von Neumann algebra* (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains $1_{\mathcal{B}(\mathcal{H})}$ and is closed in the weak operator topology.
2) If $A \subset \mathcal{B}(\mathcal{H})$ is a set, then we define the *commutant* of A by
$$A' = \{y \in \mathcal{B}(\mathcal{H})\colon xy = yx \text{ for all } x \in A\}$$
and the *bicommutant* by
$$A'' = (A')'.$$
3) A von Neumann algebra M is called a *factor* if
$$M \cap M' = \mathbb{C} \cdot 1_{\mathcal{B}(\mathcal{H})}.$$

Remark 2.2.

1) Recall that the weak operator topology WOT is the locally convex topology defined by the seminorms $p_{\xi,\eta}$ $(\xi, \eta \in \mathcal{H})$ with
$$p_{\xi,\eta}(x) = |\langle x\xi, \eta\rangle|.$$
2) Note that $M \cap M'$ is not depending on the realization on a Hilbert space (unlike the commutant), since
$$M \cap M' = Z(M) = \{x \in M\colon xy = yx \text{ for all } y \in M\}.$$

Theorem 2.3 (Bicommutant theorem)**.**
Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint subalgebra (i.e., $a \in A$ implies that also $a^* \in A$). Then A is a von Neumann algebra if and only if $A = A''$.

Remark 2.4.

1) A von Neumann algebra M is closed under the measurable functional calculus, i.e. if $x = x^* \in M$ and $f\colon \sigma(x) \to \mathbb{C}$ is a measurable bounded function, then $f(x) \in M$.
2) In particular, for $x = x^*$ and $E_\lambda(x) = \chi_{(-\infty,\lambda]}(x)$ we have by the spectral theorem
$$x = \int_{\sigma(x)} \lambda \mathrm{d}E_x(\lambda).$$
Then all spectral projections $E_x(\lambda)$ of x are contained in M. This means that there are a lot of projections in von Neumann algebras.

3) A von Neumann algebra M is closed under polar decomposition. Let $x \in M \subseteq \mathcal{B}(\mathcal{H})$, then x has a unique polar decomposition in $\mathcal{B}(\mathcal{H})$ of the form $x = u|x|$, where $|x| = \sqrt{x^*x}$ and u is a partial isometry (i.e., $u = uu^*u$) with $\ker(u) = \ker(x)$. Then it holds that $|x|, u \in M$.

4) Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a $*$-subalgebra. Then

$$M := \overline{A}^{\text{WOT}} = \overline{A}^{\text{SOT}},$$

where SOT denotes the strong operator topology, which is generated by seminorms p_ξ $(\xi \in \mathcal{H})$ with

$$p_\xi(x) = \|x\xi\|.$$

Thus the elements in the von Neumann algebra $M = A''$ can be approximated in the strong operator topology by elements of A. *Kaplansky's density theorem* says that we can also bound the norm of the elements in this approximation. (Note that this is not automatic since the norm is neither WOT- nor SOT-continuous.

Notation 2.5.
Two von Neumann algebras $M \subseteq \mathcal{B}(\mathcal{H})$, $N \subseteq \mathcal{B}(\mathcal{K})$ are called *isomorphic* if there exists a $*$-isomorphism $\Phi\colon M \to N$, in this case we write $M \cong N$.

Remark 2.6.
Commutative von Neumann algebras are classified as follows.

1) Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a separable, abelian von Neumann algebra. Then there is a separable compact Hausdorff space K and a finite Borel measure μ on K, such that

$$M \cong L^\infty(K, \mu).$$

2) Note that many $L^\infty(K, \mu)$ spaces are $*$-isomorphic. The main feature / characterization is via the number of atoms of the measure μ. More precisely we have the following possibilities (up to $*$-isomorphisms) for abelian von Neumann algebras on separable Hilbert spaces.

   a) $l^\infty(\{1, \ldots, n\})$ for $n \geq 1$ (finitely many atoms)
   b) $l^\infty(\mathbb{N})$ (infinitely many atoms)
   c) $L^\infty([0, 1])$ with respect to the Lebesgue measure (no atoms)
   d) $L^\infty([0, 1] \cup \{1, \ldots, n\})$ for $n \geq 1$ (finitely many atoms plus continuous part)
   e) $L^\infty([0, 1] \cup \mathbb{N})$ (infinitely many atoms plus continuous part)

   The number of atoms is here countable, since we restricted to separable Hilbert spaces.

Remark 2.7.

1) Any separable von Neumann algebra is isomorphic to a "direct integral" of factors, i.e. there is a family $\{M_t\}$ of factors, a measurable space T and a σ-finite measure μ on T such that

$$M \cong \int_T^{\oplus} M_t d\mu(t).$$

Thus the classification of von Neumann algebras is reduced to the classification of factors.

2) $\mathbb{C}$ is the only commutative factor. This gives all commutative von Neumann algebras via direct integrals with respect to a combination of Lebesgue and counting measure.

3) In general the classification of factors is hopeless. Thus our goal is to understand nice classes of factors.

Definition 2.8.
Let M be a von Neumann algebra. With a projection in M we will always mean an orthogonal projection, i.e., $e \in M$ with $e^* = e = e^2$.

1) Two projections $e, f \in M$ are called *equivalent* if there is a partial isometry $u \in M$, s.t. $u^*u = e$ and $uu^* = f$. In this case we write $e \sim f$.

2) Let $e, f \in M$ be projections. We write $e \preceq f$, if there is a projection $g \in M$, s.t. $e \sim g \leq f$. Equivalently, if there is a partial isometry $u \in M$ such that $u^*u = e$ and $uu^* \leq f$.

3) A projection $e \neq 0$ in M is called *minimal*, if for all projections $f \in M$ we have

$$f \leq e \implies f = 0 \text{ or } f = e.$$

4) A projection $e \in M$ is called *finite* if we have

$$e \sim f \leq e \implies f = e.$$

Definition 2.9.
Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a factor. We say

I) M is a *type* I *factor*, if there is a minimal projection in M.

II) M is a *type* II *factor*, if there is a finite but no minimal projection in M.

III) M is a *type* III *factor*, if there is no finite projection in M.

Furthermore, the case II decomposes in two subcases:

1) M is of type II_1, if $1_{\mathcal{B}(\mathcal{H})}$ is finite.

2) M is of type II_∞, if $1_{\mathcal{B}(\mathcal{H})}$ is not finite, but there are finite projections.

Remark 2.10.

1) Note that in case II_1 every projection is finite.

2) Type I factors are easy to classify.

$$M \text{ is I factor} \iff M \cong \mathcal{B}(\mathcal{H}) \text{ for a Hilbert space } \mathcal{H}.$$

Hence there is one such factor for each dimension $n = \dim \mathcal{H} \in \{\infty, 1, 2, \dots\}$; those are addressed as I_n factors.

3) Any II_∞ factor is of the form $M \otimes \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} = \infty$ and M is a II_1 factor.

4) There are many non-isomorphic II_1 factors and there is no hope for a complete classification of II_1 factors.

5) There are also a lot of III factors but those are not discussed in this class.

6) II_1 factors can also be described by the existence of a trace (plus being infinite-dimensional).

Theorem 2.11.
Let M be a factor. Then the following are equivalent.

1) M has a normal trace,

2) M has a norm continuous trace,

3) 1 is a finite projection,

4) M is of type II_1 or I_n for $n \neq \infty$.

We say then that M is a *finite* factor.

Remark 2.12.
In Theorem 2.11, note the following.

1) The characterization that 1 is a finite projection is the same as: $u^*u = 1 \implies uu^* = 1$.

2) In 4) we need $n \neq \infty$ in the type I case; for an infinite dimensional Hilbert space the identity $1_{\mathcal{B}(\mathcal{H})}$ is not finite, nor does there exist a trace on $B(\mathcal{H})$.

Definition 2.13.
Let M, N be von Neumann algebras.

1) A positive linear map $\Phi\colon M \to N$ is called *normal* if

$$\Phi(\sup_\lambda x_\lambda) = \sup_\lambda \Phi(x_\lambda)$$

holds for all increasing nets $(x_\lambda)_{\lambda\in\Lambda}$ of self-adjoint operators $x_\lambda \in M$. This means that Φ respects the lattice of projections.

2) A positive, linear functional $\tau\colon M \to \mathbb{C}$, s.t. $\tau(1) = 1$ is called a *state*. (Positive means that $\tau(x^*x) \geq 0$ for all $x \in M$.)

3) A state $\tau\colon M \to \mathbb{C}$ is called a *trace* if $\tau(xy) = \tau(yx)$ holds for all $x, y \in M$.

Theorem 2.14.
Let M be a finite factor. Then there is a unique norm-continuous trace on M. This trace is automatically normal and faithful, i.e.

$$\tau(x^*x) = 0 \implies x = 0.$$

The following proposition shows the relevance of being finite; though "finite" does not mean "finite-dimensional", elements in a finite von Neumann algebra share essential properties of matrices.

Proposition 2.15.
Let M be a factor. Then the following are equivalent.

1) M is finite.
2) Every left inverse is a right inverse: $xy = 1$ for $x, y \in M$ implies that also $yx = 1$.

Proof. First assume that property 2) holds. In order to show 1), let $1 \sim e \leq 1$, i.e. there is a partial isometry u such that $uu^* = 1$ and $u^*u = e$. This means u is a left inverse for u^*, hence by the assumption it is also a right inverse, i.e. $e = u^*u = 1$, hence 1 is finite.

Conversely assume $1 \in M$ is finite and let be $x, y \in M$ such that $xy = 1$. Let $y = u|y|$ be the polar decomposition of y. Then we have $xu|y| = 1$, and hence also $|y|(u^*x^*) = 1$. As $|y|$ has both a left and a right inverse, those must coincide and $|y|$ is invertible. By the invertibility of $|y|$ we can write $u = y|y|^{-1}$ and thus

$$u^*u = |y|^{-1}y^*y|y|^{-1} = |y|^{-2}y^*y = (y^*y)^{-1}y^*y = 1.$$

Since M is finite this yields $uu^* = 1$, hence u is invertible. So we have shown that both u and $|y|$ are invertible; hence their product y is also invertible, with $y^{-1} = |y|^{-1}u^*$. q.e.d.

Example 2.16.
Let us check some of the statements for finite factors in concrete examples. For the type I case these are of course quite concrete statements about matrices. In the type II_1 case we still need to come up with an explicit example of such a factor. It is a not obvious that such a factor does actually exist.

1) Consider the I_n factor given by $M = \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} = n < \infty$, i.e. $M = M_n(\mathbb{C})$. The fact that $M_n(\mathbb{C})$ is indeed a factor is the classical *Lemma of Schur* which says that for $M = B(\mathcal{H})$ we have $M' = \mathbb{C}1$. The trace on M is given by the normalized trace tr, i.e

$$\mathrm{tr}\colon M_n(\mathbb{C}) \to \mathbb{C}, \quad A = (a_{ij})_{i,j=1}^n \mapsto \mathrm{tr}(A) = \frac{1}{n}\mathrm{Tr}(A) = \frac{1}{n}\sum_{i=1}^n a_{ii}$$

Theorem 2.14 says that our trace tr must be faithful. Let us check this:

$$0 = \operatorname{tr}(AA^*) = \frac{1}{n}\sum_{i,j} a_{ij}\overline{a_{ij}} = \sum_{i,j} |a_{ij}|^2$$

implies $|a_{ij}| = 0$ for all $1 \leq i, j \leq n$, and hence $A = (a_{ij})_{i,j=1}^n = 0$.

2) Canonical examples of II_1 factors are given by *group factors*. Let G be a discrete group and define the *group algebra*

$$\mathbb{C}G := \left\{ \sum_{g \in G} \alpha_g g \colon \alpha_g \in \mathbb{C} \text{ and } \alpha_g \neq 0 \text{ only for finitely many } g \right\}.$$

Sometimes we write δ_g instead of g. Now we let $\mathbb{C}G$ act on itself by multiplication; this is the regular representation. In infinite dimensions we have to complete $\mathbb{C}G$ to the Hilbert space

$$l^2(G) = \left\{ \sum_{g \in G} \alpha_g \delta_g \colon \sum_{g \in G} |\alpha_g|^2 < \infty \right\},$$

with the inner product determined by

$$\langle \delta_g, \delta_h \rangle = \begin{cases} 1, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$\left\langle \sum_{g \in G} \alpha_g \delta_g, \sum_{h \in G} \beta_h \delta_h \right\rangle = \sum_{g \in G} \alpha_g \bar{\beta}_g.$$

We define the *left-regular representation* by

$$\lambda \colon \mathbb{C}G \to \mathcal{B}(l^2(G)), \quad \sum_{\text{finite}} \alpha_g \delta_g \mapsto \sum_{\text{finite}} \alpha_g \lambda(g)$$

with $\lambda(g)\delta_h = \delta_{gh}$. It holds that $\lambda(g)^* = \lambda(g^{-1})$, hence we have

$$\lambda(g)\lambda(g)^* = \lambda(g)\lambda(g^{-1}) = \lambda(e) = 1_{\mathcal{B}(\mathcal{H})};$$

in the same way we have $\lambda(g)^*\lambda(g) = 1$; so all $\lambda(g)$ are unitary operators. Thus all

$$\lambda\left(\sum_{\text{finite}} \alpha_g \delta_g \right)$$

are bounded operators on $l^2(G)$. Then we define

$$L(G) := \overline{\lambda(\mathbb{C}G)}^{\mathrm{SOT}} \subseteq \mathcal{B}(l^2(G));$$

$L(G)$ is called the *group von Neumann algebra*. Such an $L(G)$ always has a trace, which is given by (e is here the neutral element of G)

$$\tau(x) = \langle x\delta_e, \delta_e \rangle.$$

As a vector state this state is normal. Let us check the trace property. By linearity and normality it suffices to check it on group elements $g, h \in G$. For those we have

$$\begin{aligned}\tau(gh) = \langle gh\delta_e, \delta_e\rangle = \langle \delta_{gh}, \delta_e\rangle &= \begin{cases} 1 \text{ if } gh = e \\ 0 \text{ otherwise} \end{cases} \\ &= \begin{cases} 1 \text{ if } hg = e \\ 0 \text{ otherwise} \end{cases} = \langle \delta_{hg}, \delta_e\rangle = \langle hg\delta_e, \delta_e\rangle = \tau(hg).\end{aligned}$$

$L(G)$ is a factor if and only if G has *infinite conjugacy classes (icc)*, i.e. each conjugacy class $\{ghg^{-1} \colon g \in G\}$ is infinite for every $e \neq h \in G$. Thus, for an icc group G, $L(G)$ is a II_1 factor. Note that if G is finite then it is not icc and hence $L(G)$ is not a factor. In this case it decomposes into the irreducible representations of G.

Next we want to deduce the commutant $L(G)'$. Define the right-regular representation of $\mathbb{C}G$ on $l^2(G)$ by

$$\rho \colon \mathbb{C}G \to \mathcal{B}(l^2(G)), \quad \sum_{\text{finite}} \alpha_g \delta_g \mapsto \sum_{\text{finite}} \rho(\delta_g),$$

where $\rho(g)\delta_h = \delta_{hg^{-1}}$. Note that the need the inverse to have the homomorphism property of ρ:

$$\rho(g_1 g_2)\delta_h = \delta_{h(g_1g_2)^{-1}} = \delta_{hg_2^{-1}g_1^{-1}} = \rho(g_1)\rho(g_12)\delta_h.$$

We define

$$R(G) := \overline{\rho(\mathbb{C}G)}^{\text{SOT}} \subseteq \mathcal{B}(l^2(G)).$$

Then $R(G)$ and $L(G)$ commute, since

$$\lambda(g_1)\rho(g_2)\delta_h = \delta_{g_1hg_2^{-1}} = \rho(g_2)\lambda(g_1)\delta_h$$

for all $h \in G$. By linearity and continuity this implies that $\lambda(g_1)\rho(g_2) = \rho(g_2)\lambda(g_1)$ for all $g_1, g_2 \in G$. Again, by linearity and continuity this goes then also over to the group von Neuman algebras, i.e., we have $xy = yx$ for all $x \in L(G)$ and $y \in R(G)$. This means

$$L(G) \subseteq R(G)', \quad R(G) \subseteq L(G)'.$$

Actually one has equality in the inclusions above. To see this we define an anti-linear involution $J \colon l^2(G) \to l^2(G)$ given by anti-linear extension of $\delta_h \mapsto J(\delta_h) = \delta_{h^{-1}}$; i.e.

$$J\left(\sum \alpha_g \delta_g\right) = \sum \overline{a_g}\delta_{g^{-1}}.$$

Then it holds $JL(G)J = R(G)$, which implies that $L(G) = R(G)'$ and $R(G) = L(G)'$. We will come back to this in the next section, for general II_1 factors.
We know (at least in the factor case) that our trace must be faithful on $L(G)$. Let us again check this directly. Consider $x \in L(G)$ with $\tau(x^*x) = 0$. Then we have

$$0 = \tau(x^*x) = \langle x^*x\delta_e, \delta_e\rangle = \langle x\delta_e, x\delta_e\rangle$$

and thus $x\delta_e = 0$. What we need to show is that $x = 0$, i.e., $x\delta_h = 0$ for any $h \in G$. But this follows now by using the commutant $R(G)$; namely we have

$$x\delta_h = x\rho(h^{-1})\delta_e = \rho(h^{-1})x\delta_e = 0.$$

3) The above group construction gives us II_1 factors, provided we can present some i.c.c. groups. There are plenty of those. Here are two prominent examples.

   a) Consider

   $$G = S_\infty = \bigcup_{n\in\mathbb{N}} S_n \equiv \{\text{finite permutations of infinitely many points}\}.$$

   It is easy to see that this is i.c.c., thus $L(S_\infty)$ is a II_1 factor; it is called the *hyperfinite* II_1 *factor* and usually denoted by $\mathcal{R}$. In a sense, this is the "simplest" and "nicest" II_1 factor.

   b) Consider the free group $\mathbb{F}_n$ on $n \geq 2$ generators. Again it is easy to see that $\mathbb{F}_n$ is i.c.c. Thus $L(\mathbb{F}_n)$ is a II_1 factor, called the *free group factor.* Murray and von Neumann showed that $L(\mathbb{F}_n) \ncong \mathcal{R}$ (for this they introduced the so-called *property* Γ). But it s still unknown whether $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$, $n, m \geq 2$. This is the famous *free group isomorphism problem.*

3 Representation Theory for II_1 Factors: Standard Form and Left Hilbert Modules

Let M be a II_1 factor with its unique trace $\tau\colon M \to \mathbb{C}$. Via the GNS-construction w.r.t. τ we obtain a representation $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ and we put

$$L^2(M) := L^2(M,\tau) =: \mathcal{H}_\tau, \qquad \Omega := \xi_\tau.$$

Then it holds $\tau(x) = \langle x\Omega, \Omega\rangle$ for all $x \in M$. Recall that $L^2(M)$ is the completion of M w.r.t. the inner product

$$\langle x, y\rangle = \tau(y^*x).$$

(Since τ is faithful there is no kernel to divide out.) We denote the embedding of M into $L^2(M)$ by

$$M \to L^2(M), \quad x \mapsto \hat{x}.$$

The image of M under this embedding is denoted by $\widehat{M}$. Then π_τ is defined by extension of

$$\pi_\tau(x)\hat{y} = \widehat{xy}.$$

Usually we omit π_τ and just write $x\hat{y} = \widehat{xy}$. Note that we have

$$\hat{x} = \widehat{x1} = x\hat{1} = x\Omega,$$

hence the embedding is basically determined by the action of $x \in M$ on Ω.

Note that the GNS construction in general only yields that $\pi_\tau(M)$ is a C^*-algebra, but it is not clear, that it is SOT-closed in $\mathcal{B}(L^2(M)$. However this follows since τ is normal. (This statement is not obvious, but needs some characterization of "normality".)

Definition 3.1.
The representation of M on $L^2(M)$ is called the *standard representation* of M or the *standard form* of M.

Definition 3.2.
Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a subalgebra. A vector $\xi \in \mathcal{H}$ is called

1) *cyclic* if $\overline{M\xi} = \mathcal{H}$.
2) *separating* if $x \in M$ with $x\xi = 0$ implies that $x = 0$.

Proposition 3.3.
The vector $\Omega \in L^2(M)$ in the standard form is both cyclic and separating for M.

Proof. 1) The cyclicity of Ω is clear by the definition of the GNS-construction, since $M\Omega = \widehat{M}$ is dense in $L^2(M)$.

2) Now assume that $x\Omega = 0$, then $\tau(x^*x) = \langle x^*x\Omega, \Omega\rangle = 0$. Since τ is faithful this yields $x = 0$.

q.e.d.

Remark 3.4.
In some sense having a cyclic vector or a separating vector tells us something about the sizes of M resp. M'. If we have a cyclic vector then M is quite big. On the other hand if we have a separating vector then M' is quite big. This is made precise in the next proposition.

Proposition 3.5.
Let $M \subset \mathcal{B}(H)$ and $\xi \in \mathcal{H}$. Then ξ is separating for M if and only if ξ is cyclic for M'.

Proof. First assume that ξ is cyclic for M' and let $x \in M$ such that $x\xi = 0$. We have to show that $x = 0$, i.e. $x\eta = 0$ for all $\eta \in \mathcal{H}$. Consider first $\eta = y\xi \in M'\xi$ for some $y \in M'$. In this case we have

$$x\eta = xy\xi = yx\xi = 0.$$

Thus x vanishes on $M'\xi$. Since, by our assumption, this is dense in $\mathcal{H}$ it follows that $x = 0$.

Conversely assume ξ is separating for M. Denote the projection $\mathcal{H} \to \overline{M'\xi}$ onto the space $\overline{M'\xi}$ by p. We have to show that $p = 1$. First we claim $p \in M'' = M$. To see this let $x \in M'$, then we have to show that $px = xp$. By the decomposition $\mathcal{H} = \overline{M'\xi} \oplus \left(\overline{M'\xi}\right)^\perp$ it suffices to see this equality when acting on vectors of the form $y\xi + \eta$ with $y \in M'$ and $\eta \in \left(\overline{M'\xi}\right)^\perp$. Note that the action of x respects this orthogonal decomposition, i.e, we also have $x\overline{M'\xi}^\perp \subset \overline{M'\xi}^\perp$. (This follows because M' is a $*$-algebra.) Hence with $p\eta = 0$ we also have $px\eta = 0$. Now we have

$$px(y\xi + \eta) = pxy\xi + px\eta = xy\xi + px\eta = xpy\xi = xpy\xi + xp\eta = xp(y\xi + \eta).$$

Thus $xp = px$ and hence $p \in M'' = M$. From this we can now show that $p = 1_{\mathcal{B}(\mathcal{H})}$. Since ξ is separating for M and $1 - p \in M$, we only have to show that $(1 - p)\xi = 0$. But this holds (note that $1\xi \in M'\xi$):

$$(1 - p)\xi = \xi - p1\xi = \xi - \xi = 0.$$

q.e.d.

Remark 3.6.
Proposition 3.5 tells us, that in the standard form both M and M' are in some sense sufficiently large.

Example 3.7.
To get a feeling for those statements, let us onsider the corresponding situation in the I_n case, with $n < \infty$; then $M = M_n(\mathbb{C})$ and $\tau = \mathrm{tr} = \frac{1}{n}\,\mathrm{Tr}$.

1) In the defining representation $M = \mathcal{B}(\mathbb{C}^n)$ every non-trivial vector $0 \neq \xi \in \mathbb{C}^n$ is cyclic but none is separating.

2) On the other hand, for the standard representation we have

$$L^2(M_n(\mathbb{C}), \mathrm{tr}) = \mathbb{C}^{n^2} = \mathbb{C}^n \otimes \mathbb{C}^n$$

with

$$\pi_{\mathrm{tr}}(M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes 1 \subset M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$$

and

$$\pi_{\mathrm{tr}}(M_n(\mathbb{C}))' = 1 \otimes M_n(\mathbb{C}).$$

Remark 3.8.

1) Note that for the first representation in Example 3.7 the algebra $M = \mathcal{B}(\mathbb{C}^n)$ is very big and the commutant $M' = \mathbb{C}1$ very small. In this sense we have here a bad representation.

2) In the second case of Example 3.7, M and its commutant M' have the same size.

3) The crucial thing which makes the difference between the representations in Example 3.7 is the fact that the commutant M' heavily depends on the Hilbert space of the representation.

Definition 3.9.
Let $L^2(M)$ be the standard representation of M. We define the *anti-linear unitary involution* $J\colon L^2(M) \to L^2(M)$ by the extension of

$$J(x\Omega) = x^*\Omega \qquad \text{(or in alternative notation: } J\hat{x} = \widehat{x^*}.)$$

Remark 3.10.
Note that

$$J\Omega = J\hat{1} = \widehat{1^*} = \hat{1} = \Omega.$$

Theorem 3.11.
Let $M \subset \mathcal{B}(L^2(M))$ and J be as in Definition 3.9. Then it holds

$$JMJ = M'.$$

Proof. See Exercise 2. q.e.d.

Remark 3.12.
Note that the properties of J (compare Exercise 2) imply that τ is also a trace on M'. Namely, for $x, y \in M'$, we have

$$Jx\Omega = x^*\Omega, \qquad Jy^*\Omega = y\Omega.$$

Hence we have

$$\tau(xy) = \langle xy\Omega, \Omega\rangle = \langle y\Omega, x^*\Omega\rangle = \langle Jy^*\Omega, Jx\Omega\rangle = \langle x\Omega, y^*\Omega\rangle = \langle yx\Omega, \Omega\rangle = \tau(yx).$$

Thus M' is also a II_1 factor on $L^2(M)$ with trace given by the same cyclic and separating vector Ω.

Definition 3.13.
Let M be a II_1 factor. An M-*module* (or more precisely: a left M-module) is a representation of M; this means a Hilbert space $\mathcal{H}$ together with an action

$$M \times \mathcal{H} \to \mathcal{H}, \quad (x, \xi) \mapsto x\xi,$$

which is bilinear and satisfies for all $x, y \in M$ and $\xi, \eta \in \mathcal{H}$

$$x(y\xi) = (xy)\xi, \quad 1\xi = \xi, \quad \langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle$$

and is continuous in the following sense:

$$\left.\begin{array}{l}(x_n)_{n\in\mathbb{N}} \subset M,\ \|x_n\| \le 1 \\ \|x_n\|_2 \to 0 \text{ as } n \to \infty\end{array}\right\} \implies \|x_n\xi\| \to 0 \quad \forall \xi \in \mathcal{H}.$$

Recall the definition of the $\|\cdot\|_2$-norm:

$$\|x\|_2^2 := \tau(x^*x) = \tau(xx^*).$$

Example 3.14.

1) Our canonical M-module is $L^2(M)$, it has the right size to admit a cyclic and a separating vector Ω.
2) We can reduce $L^2(M)$ to “smaller” modules by “cutting down” with a projection $p \in M'$ via

   $$\mathcal{H} := pL^2(M).$$

   Note that p acts as a projection

   $$p \in M' \subset \mathcal{B}(L^2(M)).$$

   We also write $p\xi$ as ξp in bimodule language, since p commutes with everything happening on the left side; M acts from the left on $\mathcal{H}$, whereas M' acts from the right. The action of M on $\mathcal{H}$ is given by

   $$xp\xi = px\xi \qquad \text{(or maybe more natural in bimodule language: } x(\xi p) = (x\xi)p$$

   This is indeed an action; we have

   $$(xy)\xi p = xyp\xi = pxy\xi = x(py\xi) = x(yp\xi) = x(y\xi p).$$

The "size" of $L^2(M)p$ is $\tau(p)$ times the size of $L^2(M)$; also the size of M' gets smaller by a factor $\tau(p)$, but it is still a II_1 factor; M' gets replaced by $pM'p$.
For instance consider

$$M = M_2(\mathbb{C}), \quad L^2(M_2(\mathbb{C}) = \mathbb{C}^4 \equiv M_2(\mathbb{C}).$$

We can reduce this representation by a projection $p \in M' = M_2(\mathbb{C})$ of size $\tau(p) = 1/2$ to a representation of $M = M_2(\mathbb{C})$ on $\mathbb{C}^2$. But there is no projection in M' to reduce it to $\mathbb{C}^3$ or $\mathbb{C}$.
Note that if we cut down the standard representation $L^2(M)$ by a projection $p \in M'$ to $\mathcal{H} = pL^2(M)$, then we loose our separating vector; Ωp is not separating any more. Namely put $q := JpJ \in M$; note that

$$q\Omega = JpJ\Omega = Jp\Omega = p^*\Omega = p\Omega.$$

Then we have

$$(1-q)\Omega p = (1-q)p\Omega = p\Omega - q(q\Omega) = p\Omega - q^2\Omega = p\Omega - q\Omega = p\Omega - p\Omega = 0.$$

Thus there is $0 \neq 1 - q \in M$ such that $(1-q)\Omega p = 0$, i.e. Ωp is not separating for M.

3) We can amplify $L^2(M)$ to "bigger" modules by taking direct sums

$$\mathcal{H} = \underbrace{L^2(M) \oplus \cdots \oplus L^2(M)}_{n\text{-times}}$$

with the (entrywise) action

$$x(\xi_1 \oplus \cdots \oplus \xi_n) = (x\xi_1 \oplus \cdots \oplus x\xi_n).$$

This $\mathcal{H}$ is somehow n-times bigger than $L^2(M)$. But we loose our cyclic vector by this amplification. Consider $\omega = \Omega \oplus \cdots \oplus \Omega \in \mathcal{H}$, then we have

$$\overline{M\omega} = \operatorname{span}\{\xi \oplus \cdots \oplus \xi \colon \xi \in L^2(M)\} \neq \mathcal{H}.$$

Note that now also the commutant is getting bigger, M' is getting replaced by $M_n(M') = M_n(\mathbb{C}) \otimes M'$, but this is still a II_1 factor.

4) There is an even "bigger" representation, an infinite amplification of $L^2(M)$ by taking infinitely many direct summands (in the Hilbert space sense), i.e.

$$\mathcal{H} = l^2(L^2(M)) \cong l^2(\mathbb{N}) \otimes L^2(M)$$

where

$$l^2(L^2(M)) = \left\{ (\xi_n)_{n\in\mathbb{N}} \colon \xi_n \in L^2(M), \ \sum_{n\in\mathbb{N}} \|\xi_n\|^2 < \infty \right\}$$

and the action is defined via

$$x(\xi_n)_{n\in\mathbb{N}} = (x\xi_n)_{n\in\mathbb{N}}.$$

We get now as commutant $M' \otimes \mathcal{B}(l^2(\mathbb{N}))$. Since we have no trace on $\mathcal{B}(l^2(\mathbb{N}))$ this commutant is not a II_1 factor, but II_∞.

5) One can combine 2) and 3) to get

$$\mathcal{H} = \underbrace{L^2(M) \oplus \cdots \oplus L^2(M)}_{n\text{-times}} \oplus L^2(M)p$$

with $p \in M'$. Then $\mathcal{H}$ is "$(n + \tau(p))$-times" as big as $L^2(M)$.

6) By the previous examples we have for each $\alpha \in [0, \infty]$ an M-module which is in some sense α-times as big as $L^2(M)$.

This raises the following two questions.
- Are those modules different for different α?
- Are there modules which are not of this form?

For the purpose of answering these questions we need a notion to compare M-modules, hence we give the following definition.

Definition 3.15.
Let M be a II_1 factor. Then two modules $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *(unitarily) equivalent*, $\mathcal{H}_1 \cong \mathcal{H}_2$, if the representations are unitary equivalent, i.e., there is an unitary $u\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$u(x\xi) = x(u\xi) \qquad \text{for all } \xi \in \mathcal{H}_1 \text{ and } x \in M.$$

This means u commutes with the action of M according to the following diagram.

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{u} & \mathcal{H}_2 \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ \mathcal{H}_1 & \xrightarrow{u} & \mathcal{H}_2 \end{array}$$

Theorem 3.16.
Let M be a II_1 factor and $\mathcal{H}$ a separable M-module. Then there is an isometry

$$v\colon \mathcal{H} \to L^2(M) \otimes l^2(\mathbb{N})$$

such that

$$vx = (x \otimes 1)v \qquad \text{for all } x \in M.$$

Furthermore we have

$$vv^* \in (M \otimes 1)' \subset \mathcal{B}(L^2(M) \otimes l^2(\mathbb{N}))$$

and for the trace $\mathrm{tr} = \tau \otimes \mathrm{Tr}$ on the II_∞ factor $(M \otimes 1)' = M' \otimes \mathcal{B}(l^2(\mathbb{N}))$ the number $\mathrm{tr}(vv^*) = \tau \otimes \mathrm{Tr}(vv^*)$ is independent of v. (Note that on a II_∞ factor we cannot normalize our tracial state tr in the usual way since $\mathrm{tr}(1 \otimes 1) = \infty$; however, it is normalized in the sense that $\mathrm{tr}(1 \otimes q) = 1$ for any rank 1 projection q in $\mathcal{B}(l^2(\mathbb{N}))$.)

Definition 3.17.
Let M be a II_1 factor and $\mathcal{H}$ be a M-module. We define the *M-dimension* or the *coupling constant* of $\mathcal{H}$ by the number

$$\dim_M \mathcal{H} := \mathrm{tr}(vv^*) \in [0, \infty]$$

from Theorem 3.16.

Remark 3.18.

1) The assertion of Theorem 3.16 is that any action of M on some $\mathcal{H}$ is equivalent to $p(L^2(M) \otimes l^2(\mathbb{N}))$ for some $vv^* = p \in (M \otimes 1)'$ and $\dim_M \mathcal{H}$ is then the trace of this projection p in $(M \otimes 1)'$. The latter is II_∞ , so $\mathrm{tr}(p)$ can take on all values in $[0, \infty]$.
2) It was called "coupling constant" by Murray and von Neumann as it compares the sizes of M and M' in the given representation. Actually, they defined it as the ratio

$$\frac{\mathrm{tr}_M(q)}{\mathrm{tr}_{M'}(p)},$$

where q is the projection onto the space $\overline{M'\xi}$ and p is the projection onto the space $\overline{M\xi}$, for an arbitrary $0 \neq \xi \in \mathcal{H}$. Note that $p \in M'$ and $q \in M$. The idea is that the numbers $\tau_M(q), \tau_{M'}(p)$ measure how close ξ is to be cyclic for M resp. M' and this ratio is equal for every ξ. However, this independence of the ratio from the choice of the ξ is a quite non-trivial fact, which is due to Murray and von Neumann. (Our approach avoids to address this question.) Given this result by Murray and von Neumann one sees for an M-module $\mathcal{H}$:

$$\begin{aligned} \dim_M \mathcal{H} \leq 1 &\iff M \text{ has a cyclic vector,} \\ \dim_M \mathcal{H} \geq 1 &\iff M \text{ has a separating vector,} \\ \dim_M \mathcal{H} = 1 &\iff M \text{ has a cyclic and separating vector.} \end{aligned}$$

Note that the directions " $\Longrightarrow$ " follow from the fact that we know that reductions or amplifications of Ω will still be cyclic or separating, respectively. The other direction, however, is not clear in our approach. We know that the reductions or amplifications of Ω loose the property of being separating or cyclic, respectively; but we do not know that there could not be other cyclic or separating vectors.

Proof of Theorem 3.16. Consider the M-module

$$\mathcal{K} = \mathcal{H} \oplus l^2(L^2(M))$$

with the canonical action of M on the second summand and let M' be the commutant of M in $\mathcal{B}(\mathcal{K})$; M' is a II_∞ factor. Consider the projections $p = 1 \oplus 0$ and $q = 0 \oplus 1$. Both are clearly in M'. Since q is an infinite projection in M' and M' is a factor, we have, by the comparision theory for projections in factors, that any projection in M', in particular also p, is equivalent to a subprojection of q; hence we have $p \preceq q$, i.e. there is a partial isometry $u \in M'$ such that $u^*u = p$ and $uu^* \leq q$. We write

$$u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{B}(\mathcal{K}) = \begin{bmatrix} \mathcal{B}(\mathcal{H}) & \mathcal{B}(l^2(L^2(M)), \mathcal{H}) \\ \mathcal{B}(\mathcal{H}, l^2(L^2(M))) & \mathcal{B}(l^2(L^2(M))) \end{bmatrix}$$

and thus

$$u^* = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = p = u^*u = \begin{bmatrix} * & * \\ * & b^*b + d^*d \end{bmatrix},$$

hence we have $b^*b + d^*d = 0$ and thus $b = d = 0$. Furthermore we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = q \geq uu^* = \begin{bmatrix} aa^* + bb^* & * \\ * & * \end{bmatrix}$$

and thus, since the diagonals preserve the order, $aa^* + bb^* = 0$; hence also $a = 0$. Hence we have

$$u = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \qquad \text{and thus} \qquad uu^* = \begin{bmatrix} 0 & 0 \\ 0 & cc^* \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = q$$

and thus $cc^* \leq 1$. Moreover it holds

$$u^*u = \begin{bmatrix} c^*c & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = p$$

or equivalently $c^*c = 1$. Hence

$$v := c \colon \mathcal{H} \to L^2(M) \otimes l^2(\mathbb{N})$$

is an isometry. With respect to the direct sum decomposition of $\mathcal{K}$ we have the action

$$M \to \mathcal{B}(\mathcal{K}), \quad x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \otimes 1 \end{bmatrix}$$

and since $u \in M'$ it holds

$$\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \otimes 1 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \otimes 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix},$$

which means we have $vx = (x \otimes 1)v$ for every $x \in M$. Note that since M' is a $*$-algebra it holds

$$uu^* = \begin{bmatrix} 0 & 0 \\ 0 & vv^* \end{bmatrix} \in M', \qquad \text{i.e., } vv^* \in (M \otimes 1)' \subset \mathcal{B}(L^2(M) \otimes l^2(\mathbb{N})).$$

Thus we can take the II_∞ trace $\mathrm{tr}(vv^*)$. Note that neither v nor v^* are in $(M \otimes 1)'$, hence $\mathrm{tr}(vv^*) \neq \mathrm{tr}(v^*v)$.

Let now $w \colon \mathcal{H} \to L^2(M) \otimes l^2(\mathbb{N})$ be another isometry with $wx = (x \otimes 1)w$ for every $x \in M$. We have

$$\begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \in M'$$

and thus

$$\begin{bmatrix} 0 & 0 \\ 0 & vw^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix} \in M' \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ 0 & wv^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}^* \begin{bmatrix} 0 & 0 \\ w & 0 \end{bmatrix} \in M'.$$

This implies $vw^*, wv^* \in (M \otimes 1)'$. Since $vv^* = vw^*wv^*$ and $wv^*vw^* = ww^*$, it holds

$$\mathrm{tr}(vv^*) = \mathrm{tr}((vw^*)(wv^*)) = \mathrm{tr}((wv^*)(vw^*)) = \mathrm{tr}(ww^*),$$

where we have used the tracial property of tr on $(M \otimes 1)'$. q.e.d.

Theorem 3.19.
Let M be a II_1 factor. Then $\dim_M$ has the following properties

1) $\dim_M \mathcal{H} \in [0, \infty]$ and all values occur.
2) It holds

$$\dim_M \mathcal{H} < \infty \iff M' \text{ is } \mathrm{II}_1 \text{ factor},$$
$$\dim_M \mathcal{H} = \infty \iff M' \text{ is } \mathrm{II}_\infty \text{ factor}.$$

3) It holds

$$\dim_M \mathcal{H} = \dim_M \mathcal{K} \iff \mathcal{H} \cong \mathcal{K}.$$

4) Let I be a countable index set, then

$$\dim_M(\bigoplus_{i \in I} \mathcal{H}_i) = \sum_{i \in I} \dim_M \mathcal{H}_i.$$

5) It holds

$$\dim_M(L^2(M)q) = \tau(q)$$

for all projections $q \in M'$.

6) For a projection $p \in M$, we have

$$\dim_{pMp}(p\mathcal{H}) = \frac{1}{\tau(p)} \dim_M \mathcal{H}.$$

Moreover if M' is a II_1 then we have

7) For projections $p \in M'$ we have

$$\dim_M(\mathcal{H}p) = \tau_{M'}(p) \dim_M \mathcal{H} \qquad \text{(note that } Mp \cong M\text{)}.$$

8) It holds

$$\dim_{M'} \mathcal{H} = \frac{1}{\dim_M \mathcal{H}}.$$

Proof. See Exercise 5. q.e.d.

4 Jones Index: Definition and Elementary Properties

Definition 4.1.
Let $N \subset M$ be II_1 factors. The number

$$[M:N] := \dim_N L^2(M)$$

is called the *(Jones) index* of N in M.

Remark 4.2.

1) Note that $N \subset M$ acts on $L^2(M)$.
2) We have $L^2(N) \subset L^2(M)$, hence

$$L^2(M) = L^2(N) \oplus L^2(N)^\perp$$

and thus

$$\dim_N L^2(M) = \underbrace{\dim_N L^2(N)}_{=1} + \underbrace{\dim_N L^2(N)^\perp}_{\geq 0}.$$

Note that we have equality in the second term if and only if $L^2(N) = L^2(M)$, i.e. $N = M$. So we have in general $[M:N] \geq 1$ and $[M:N] = 1$ if and only if $N = M$.

Proposition 4.3.
If $N \subset M$ is acting on $\mathcal{H}$ such that $\dim_N(\mathcal{H}) < \infty$, then

$$[M:N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.$$

Proof. By taking direct sums, we can assume that $\dim_M \mathcal{H} \geq 1$. Note $N \subset M$ implies $M' \subset N'$. By assumption N' is a II_1 factor, hence M' is a II_1 factor. Let $p \in M'$ with

$$\tau_{M'}(p) = \frac{1}{\dim_M \mathcal{H}},$$

then it holds

$$\dim_M(\mathcal{H}p) = \tau_{M'}(p) \dim_M \mathcal{H} = 1.$$

Thus we have $\mathcal{H}p \cong L^2(M)$ and hence

$$[M:N] = \dim_N L^2(M) = \dim_N(\mathcal{H}p) = \tau_{N'}(p) \dim_N \mathcal{H} = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}}.$$

Note that we have used the fact that because of the uniqueness of traces on factors we have $\tau_{N'}|_{M'} = \tau_{M'}$; so that we have for $p \in M' \subset N'$

$$\tau_{N'}(p) = \tau_{M'}(p) = \frac{1}{\dim_M \mathcal{H}}.$$

q.e.d.

Proposition 4.4.
Let $N \subset M$ be II_1 factors and assume $[M : N] < \infty$. Let p, q be projections.

1) If $0 \neq p \in N \subset M$, then $pNp \subset pMp$ are also II_1 factors and we have
$$[pMp : pNp] = [M : N].$$

2) If $0 \neq q \in M' \subset N'$, then $Nq \subset Mq$ are also II_1 factors and we have
$$[Mq : Nq] = [M : N].$$

3) If $0 \neq p \in N' \cap M$, then $pNp = Np \subset pMp$ are also II_1 factors and
$$[pMp : Np] = \tau_M(p) \cdot \tau_{N'}(p) \cdot [M : N].$$
Note that $N' \cap M$ is not necessarily a factor, thus we have no uniqueness argument for the trace and the values $\tau_M(p)$ and $\tau_{N'}(p)$ can be different.

Proof. This follows from Proposition 4.3 and the formulas for $\dim_M \mathcal{H}$ from Theorem 3.19. Let us only do the third part:
$$[pMp : Np] = \frac{\dim_{Np} p\mathcal{H}}{\dim_{pMp} p\mathcal{H}} = \frac{\tau_{N'}(p) \cdot \dim_N \mathcal{H}}{\tau_M(p)^{-1} \cdot \dim_M \mathcal{H}} = \tau_M(p) \cdot \tau_{N'}(p) \cdot [M : N].$$
q.e.d.

Corollary 4.5.
Let $N \subset M$ be II_1 factors with $[M : N] < \infty$. Let $0 \neq p_i$, $i \in I$ be projections with
$$p_i \in N' \cap M \qquad \text{and} \qquad \sum_{i \in I} p_i = 1.$$
Then we have
$$[M : N] = \sum_{i \in I} \frac{1}{\tau_M(p_i)} [p_i M p_i : N p_i]$$
and hence
$$[M : N] \geq \sum_{i \in I} \frac{1}{\tau_M(p_i)}.$$

Proof. By Proposition 4.4, we have for each $i \in I$
$$\tau_{N'}(p_i) \cdot [M : N] = \frac{1}{\tau_M(p_i)} [p_i M p_i : N p_i].$$
By summing over i and noting that
$$\sum_i \tau_{N'}(p_i) = \tau_{N'}(\sum_i p_i) = \tau_{N'}(1) = 1$$
we have the first assertion. The inequality follows by the fact that
$$[p_i M p_i : N p_i] \geq 1 \quad \text{for all } i \in I.$$
q.e.d.

Corollary 4.6.
Let $N \subset M$ be II_1 factors. Then we have the following.

1) If $[M : N] < \infty$, then we have $\dim(N' \cap M) < \infty$.
2) If $[M : N] < 4$, then we have $N' \cap M = \mathbb{C}1$.

Proof.

1) If $N' \cap M$ is infinite dimensional, then we find infinitely many orthogonal projections $p_i \in N' \cap M$ which sum up to 1. Hence it holds
$$[M : N] \geq \sum_{i=1}^{\infty} \underbrace{\frac{1}{\tau_M(p_i)}}_{\geq 1} = \infty.$$
2) If $N' \cap M \neq \mathbb{C}1$, then we find a projection $p \neq 0, 1$ with $p \in N' \cap M$. Hence we have $p + (1 - p) = 1$ and by, Corollary 4.5, we get
$$[M : N] \geq \frac{1}{\tau_M(p)} + \frac{1}{\tau_M(1-p)} = \frac{1}{\tau_M(p)} + \frac{1}{1 - \tau_M(p)} \geq 4.$$
Note that
$$\frac{1}{t} + \frac{1}{1-t} \geq 4 \quad \text{for all } t \in (0, 1).$$

q.e.d.

Definition 4.7.
Let $N \subset M$ be II_1 factors. We call the subfactor $N \subset M$ *irreducible* if $N' \cap M = \mathbb{C}1$.

Example 4.8.

1) Let N be a II_1 factor and
$$M = M_k(N) \cong M_k(\mathbb{C}) \otimes N$$
for $k \geq 2$. Then consider
$$N \cong 1 \otimes N \subset M_k(\mathbb{C}) \otimes N = M,$$
i.e. $N \subset M$ is a subfactor via $x \mapsto 1 \otimes x$. But it is reducible, since
$$N' \cap M \cong M_k(\mathbb{C}) \otimes 1.$$
By Corollary 4.6 we must have $[M : N] \geq 4$. Actually we can calculate the value of the index explicitly:
$$L^2(M) = L^2(M_k(\mathbb{C})) \otimes L^2(N) = \bigoplus_{k^2\text{-times}} L^2(N),$$
so that $\dim_N L^2(M) = k^2$ and thus
$$[M : N] = k^2 \in \{4, 9, 16, 25 \dots\}.$$

2) One can make the estimate in the proof of Corollary 4.6(2) to an equality as follows. Take $M = \mathcal{R}$, the hyperfinite II_1 factor, and $p \in \mathcal{R}$ a projection with $\tau(p) = t$ for arbitrary $t \in (0,1)$. A results of Murray and von Neumann states that

$$p\mathcal{R}p \cong \mathcal{R} \qquad \text{and} \qquad (1-p)\mathcal{R}(1-p) \cong \mathcal{R}.$$

This means that there is a $*$-isomorphism

$$\theta \colon p\mathcal{R}p \to (1-p)\mathcal{R}(1-p).$$

Put $M := \mathcal{R}$ and

$$N := \{x + \theta(x) \colon x \in p\mathcal{R}p\} \subset p\mathcal{R}p + (1-p)\mathcal{R}(1-p) \subset M.$$

Note that $N \cong p\mathcal{R}p \cong \mathcal{R}$. We have $p \in M$, $p = 1 \oplus 0 \in N'$, $pMp = Np$, and

$$(1-p)M(1-p) = \theta(pMp) = N(1-p).$$

Thus

$$\begin{aligned} [M:N] &= \frac{1}{\tau(p)}[pMp : Np] + \frac{1}{\tau(1-p)}[(1-p)M(1-p) : N(1-p)] \\ &= \frac{1}{\tau(p)} + \frac{1}{1-\tau(p)}. \end{aligned}$$

Note that the map

$$(0,1) \to [4,\infty), \quad t \mapsto \frac{1}{t} + \frac{1}{1-t}$$

is surjective. Hence each value in $[4,\infty)$ shows up as a possible index in the case where $M \cong N \cong \mathcal{R}$ is the hyperfinite II_1 factor. (Note as a side remark that by a non-trivial result of Connes all subfactors of $\mathcal{R}$ must itself be $\mathcal{R}$.)

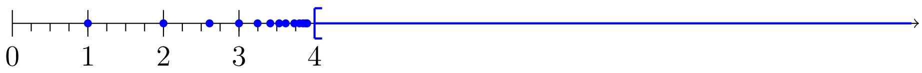
There is no obvious way to construct subfactors with indices smaller than 4. The clarification of what happens there is the following big result of Vaughan Jones. It is one of our main goals to prove this in the next sections.

Theorem 4.9 (Jones 1983)**.**
Let $N \subset M$ be II_1 factors. If $[M:N] < 4$, then

$$[M:N] \in \{4\cos^2(\frac{\pi}{n+2}) \colon n = 1,2,3\dots\},$$

i.e. the values of index less than 4 accumulate at 4 and have gaps. Furthermore all possible values $4\cos^2(\frac{\pi}{n+2})$ show up as indices of hyperfinite subfactors.

n	$[M : N]$
1	1
2	2
3	2.618
4	3
5	3.247
6	3.414
$\vdots$	$\vdots$



5 Conditional Expectation and Basic Construction

Definition 5.1.
Let M be a finite von Neumann algebra with faithful normal trace τ and $N \subset M$ a von Neumann subalgebra, with $1_N = 1_M$. A *conditional expectation* $E\colon M \to N$ is a linear and positive map with the properties

1) $E(b) = b$ for all $b \in N$,
2) $E(b_1 x b_2) = b_1 E(x) b_2$ for all $b_1, b_2 \in N, x \in M$.

Theorem 5.2 (Umegaki 1954)**.**
Let M be a finite von Neumann algebra with faithful normal trace and $N \subset M$ a von Neumann subalgebra. Then there exists a unique conditional expectation $E\colon M \to N$ such that $\tau \circ E = \tau$.

Proof. The idea is to embed everything in $L^2(M, \tau)$, then E corresponds to the orthogonal projection $L^2(M, \tau) \to L^2(N, \tau)$. So let us consider $L^2(M, \tau)$, then $L^2(N, \tau) \subset L^2(M, \tau)$ is a sub-Hilbert space; we denote by $e\colon L^2(M) \to L^2(N)$ the corresponding orthogonal projection in $\mathcal{B}(L^2(M))$. We denote the restriction of e to M (where as usual $M \cong \widehat{M} \subset L^2(M)$) by

$$\tilde{E}\colon M \to L^2(N), \quad x \mapsto \tilde{E}(x) = e\hat{x} = ex\Omega \in L^2(N).$$

We want to show that $\tilde{E}(x) \in \widehat{N}$, i.e. that it is of the form $\tilde{E}(x) = E(x)\Omega$ for $E(x) \in N$. Note that $exe\Omega = ex\Omega$, hence we expect exe and $E(x)$ to agree on $L^2(N)$; i.e., we should have $exe = E(x)e$.

In order to get this $E(x)$, let $J\colon L^2(M) \to L^2(M)$ be the usual anti-linear involution on $L^2(M)$ given by the extension of $J(x\Omega) = x^*\Omega$. (We defined this J actually only in the case of a II_1 factor; but the general finite case works in the same way.) Note that this restricts to $J\colon L^2(N) \to L^2(N)$. and we have $e(JNJ)e = eN'e$. We will show, that

$$exe \in e(JNJ)'e = eN''e = eNe = Ne,$$

from which it follows that $exe = E(x)e$ for some $E(x) \in N$. Note that then $E(x)$ is uniquely determined, since

$$E(x)\Omega = E(x)e\Omega = exe\Omega = ex\Omega$$

and Ω is separating for M (and hence for N).

Let us now show that exe commutes with JNJ. For this consider $x \in M$ and $y \in N$, then we have (note that $(exe)(eJyJe) = exeJyJ$)

$$(exe)(JyJ) = exJyJe = eJ \underbrace{JxJ}_{\in M' \subset N'} yJe = eJyJxJJe = eJyJxe = (JyJ)(exe).$$

Hence $E\colon M \to N$ is uniquely defined by $E(x)e = exe$.

The map E is clearly linear and positive and satisfies:

- $E(y) = y$ for all $y \in N$
  since then $eye = ey$
- $E(y_1 x y_2) = y_1 E(x) y_2$ for all $y_1, y_2 \in N$ and $x \in M$
  since $y_1 E(x) y_2 e = e y_1 x y_2 e$
- $\tau \circ E = \tau$
  since $\tau(E(x)) = \langle E(x)\Omega, \Omega\rangle = \langle E(x)e\Omega, \Omega\rangle = \langle exe\Omega, \Omega\rangle = \langle x\Omega, \Omega\rangle = \tau(x)$.

Hence $E\colon M \to N$ is a conditional expectation with $\tau \circ E = \tau$.

Finally we have to show the uniqueness. Assume there is another conditional expectation $\tilde{E}\colon M \to N$, which preserves the trace τ, i.e. $\tau \circ \tilde{E} = \tau$. Let $x \in M$, then we want to show that $E(x) = \tilde{E}(x)$, for which it suffices to show that $E(x)\Omega = \tilde{E}(x)\Omega$. Now, for $y \in N$, we have

$$\begin{aligned}\langle \tilde{E}(x)\Omega, y\Omega\rangle = \tau(y^*\tilde{E}(x)) = \tau(\tilde{E}(y^*x)) &= \tau(y^*x) \\ &= \tau(E(y^*x)) = \tau(y^*E(x)) = \langle E(x)\Omega, y\Omega\rangle,\end{aligned}$$

hence

$$(\tilde{E}(x) - E(x))\Omega \perp \widehat{N}.$$

Since $(\tilde{E}(x) - E(x))\Omega \in \hat{N}$ this implies that $(\tilde{E}(x) - E(x))\Omega = 0$, and hence $E(x) = \tilde{E}(x)$ for all $x \in M$.

As pointed out by Dietmar Bisch a much shorter proof of the uniqueness follows by just observing that a trace-preserving conditional expectation has to be the restriction of the orthogonal projection from $L^2(M)$ to $L^2(N)$. q.e.d.

Remark 5.3.
Note that E satisfies also

1) $E(x^*) = E(x)^*$ for all $x \in M$
2) for all $x \in M$: $E(x^*x) = 0 \implies x = 0$.
3) For all $x \in M$, $E(x)$ is uniquely determined by the requirements that $E(x) \in N$ and
$$\tau((E(x) - x)y) = 0 \qquad \text{for all } y \in N.$$

This can be seen as follows.

1) is actually true for any positive map between C^*-algebras. Namely, if E maps positive to positive, then it maps also selfadjoints to selfadjoints (since those can be written as the difference of two positive operators). But any element can be written as $x + iy$, with $x^* = x$ and $y^* = y$, and for those one has
$$E((x+iy)^*) = E(x - iy) = E(x) - iE(y) = (E(x) + iE(y))^* = (E(x+iy))^*.$$
2) This follows from the faithfulness of τ. Assume $E(x^*x) = 0$. Then
$$\tau(x^*x) = \tau(E(x^*x)) = 0,$$
and hence $x = 0$.

3) This is just the fact that in the L^2-space $E(x)$ is given by the orthogonal projection onto $L^2(N)$, hence it is determined by the condition that $E(x) - x$ must be orthogonal to $L^2(N)$; for which it suffices to have orthogonality for the dense subset $\hat{N} \subset L^2(N)$.

Proposition 5.4.
Let $N \subset M$ be II_1 factors, $e \in \mathcal{B}(L^2(M))$ the orthogonal projection onto $L^2(N)$ and $E\colon M \to N$ the conditional expectation from Theorem 5.2. Then we have the following in $\mathcal{B}(L^2(M))$.

1) $E(x)$ is for $x \in M$ uniquely determined by $exe = E(x)e$.
2) For $x \in M$ we have: $xe = ex$ if and only if $x \in N$.
3) $N' = (M' \cup \{e\})''$.

Proof.

1) This was shown in the proof of Theorem 5.2.
2) For the implication "$\Leftarrow$" take $x \in N$. Then we have $xe = E(x)e = exe$, and in the same way for $x^* \in N$, $x^*e = ex^*e$. Thus

$$ex = (x^*e)^* = (ex^*e)^* = exe = xe.$$

Conversely, assume $ex = xe$. Then we have

$$E(x)e = exe = xe^2 = xe$$

and hence

$$(E(x) - x)\Omega = (E(x) - x)e\Omega = 0.$$

Since Ω is separating for M we have $x = E(x) \in N$.
3) By 2) we have

$$N = M \cap \{e\}' = M'' \cap \{e\}' = (M' \cup \{e\})'$$

and hence

$$N' = (M' \cup \{e\})''.$$

q.e.d.

Remark 5.5.
Note that $N \subset M$ implies $M' \subset N'$; thus the previous proposition tells us that the "difference" between M' and N' is given by the projection e.

Definition 5.6.
Let $N \subset M$ be II_1 factors and denote the orthogonal projection onto $L^2(N)$ by

$$e_N \colon L^2(M) \to L^2(N) \subset L^2(M)$$

We put

$$M_1 = \{M \cup \{e_N\}\}'' \subseteq \mathcal{B}(L^2(M)).$$

(Note that M_1 is the von Neumann algebra generated by M and e_N in $\mathcal{B}(L^2(M))$.) We write $M_1 = \langle M, e_N \rangle$ and call M_1 the *basic construction* for $N \subset M$.

Remark 5.7.
Later we will iterate this basic construction and build up towers in this way: $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \dots$. For now, we first investigate this bacic construction.

Proposition 5.8.
Let $N \subset M$ be II_1 factors and $M_1 = \langle M, e_N \rangle$ the basic construction. Then we have:

1) $M + Me_N M$ is a weakly dense $*$-subalgebra of M_1.
2) $M_1 = JN'J$.
3) $[M : N] < \infty$ if and only if M_1 is a II_1 factor.
   In this case $M \subset M_1$ is a II_1 subfactor and we have
$$[M_1 : M] = [M : N].$$
4) $e_N M_1 e_N = N e_N$.

For the following assume that $[M : N] < \infty$, so that M_1 is a II_1 factor with unique trace τ_{M_1} and trace-preserving conditional expectation $E_M : M_1 \to M$.

5) $\tau_{M_1}(e_N) = [M : N]^{-1}$
6) We have for all $x \in M$
$$\tau_{M_1}(xe_N) = \tau_{M_1}(e_N) \cdot \tau(x) = \frac{\tau(x)}{[M : N]}.$$
7) $E_M(e_N) = \tau_{M_1}(e_N) \cdot 1_{\mathcal{B}(L^2(M))} = [M : N]^{-1} \cdot 1_{\mathcal{B}(L^2(M))}$.

Proof.

1) $M + Me_N M$ is closed under multiplication, since we have $e_N x e_N = E_N(x) e_N$; it contains M and e_N, hence it is dense.q
2) In the following we will often just write e for e_N. By Proposition 5.4 we have $N' = \mathrm{vN}(M', e)$, then
$$JN'J = \mathrm{vN}(JM'J, JeJ) = M_1,$$

because we have $JM'J = M$ and $JeJ = e$; the latter follows since we have for all $x \in M$:

$$\begin{aligned} JeJx\Omega = Jex^*\Omega = Jex^*e\Omega = JE(x^*)\Omega \\ = JE(x)^*\Omega = E(x)\Omega = E(x)e\Omega = exe\Omega = ex\Omega. \end{aligned}$$

3) We have $[M : N] < \infty$ if and only if N' is a II_1 factor (by Theorem 3.19); which is equivalent to the fact that $M_1 = JN'J$ is a II_1 factor. We have then

$$\begin{aligned} [M_1 : M] = \frac{\dim_M L^2(M)}{\dim_{M_1} L^2(M)} &= \frac{1}{\dim_{M_1} L^2(M)} \\ &= \frac{1}{\dim_{JN'J} L^2(M)} \\ &= \frac{1}{\dim_{N'} L^2(M)} = \dim_N L^2(M) = [M : N]. \end{aligned}$$

4) This follows from 1) and $e_N x e_N = E_N(x) e_N$; note in particular

$$e_N M e_N M e_N = \underbrace{e_N M e_N}_{E_N(M)e_N} \underbrace{e_N M e_N}_{E_N(M)e_N} e_N = \underbrace{E_N(M) E_N(M)}_{\in N} e_N.$$

5) Since $e_N \in N'$, we have

$$\begin{aligned} 1 &= \dim_N L^2(N) \\ &= \dim_N e_N L^2(M) \\ &= \tau_{N'}(e_N) \cdot \dim_N(L^2(M)) \\ &= \tau_{N'}(e_N) \cdot [M : N], \end{aligned}$$

and since $M_1 = JN'J$ and $JeJ = e$ this yields

$$\tau_{M_1}(e) = \tau_{JN'J}(JeJ) = \tau_{N'}(e) = [M : N]^{-1}.$$

6) For $y_1, y_2 \in N$ we have

$$\tau_{M_1}(y_1 y_2 e_N) = \tau_{M_1}(y_2 e_N y_1) = \tau_{M_1}(y_2 y_1 e_N),$$

since $e_N \in N'$, i.e. $y_1 \in N$ implies $y_1 e_N = e_N y_1$. This shows that the the positive linear map $y \mapsto \tau_{M_1}(y e_N)$ is tracial on N; since the trace on a II_1 factor is unique we must have

$$\tau_{M_1}(y e_N) = c \cdot \tau_N(y) = c \cdot \tau(y),$$

where $y = 1$ implies that

$$c = \tau_{M_1}(e_N) = [M : N]^{-1}.$$

Consider now general $x \in M$, then

$$\tau_{M_1}(x e_N) = \tau_{M_1}(e_N x e_N) = \tau_{M_1}(E_N(x) e_N) = \tau(E_N(x)) \cdot c = \tau(x) \cdot c.$$

7) By 6), we have for all $x \in M$ (note that then $\tau_{M_1}(x) = \tau(x)$)

$$\tau_{M_1}((e_N - \tau_{M_1}(e_N) \cdot 1)x) = \tau_{M_1}(e_N x) - \tau_{M_1}(e_N) \cdot \tau_{M_1}(x) = 0.$$

Since $\tau_{M_1}(e_N) \cdot 1 \in M$, this implies then, by Remark 5.3, that $E_M(e_N) = \tau_{M_1}(e_N) \cdot 1$.

q.e.d.

Proposition 5.9.
Let $N \subset P \subset M$ be II_1 factors with $[M : N] < \infty$. Then we have

$$[M : N] = [M : P] \cdot [P : N].$$

Proof. It is an exercise to check that $[M : N] < \infty$ implies $[M : P] < \infty$ and $[P : N] < \infty$. Then we have

$$[M : N] = \frac{\dim_N \mathcal{H}}{\dim_M \mathcal{H}} = \frac{\dim_N \mathcal{H}}{\dim_P \mathcal{H}} \cdot \frac{\dim_P \mathcal{H}}{\dim_M \mathcal{H}} = [M : P] \cdot [P : N].$$

q.e.d.

Proposition 5.10.
There is no subfactor $N \subset M$ with $1 < [M : N] < 2$.

Proof. Let $N \subset M$ with $[M : N] \neq 1$. Then $M \neq N$ and $e_N \neq 1$. Put $\lambda := [M : N]^{-1}$. We have by the basic construction $N \subset M \subset M_1$ with

$$[M : N] = [M_1 : M] = \lambda^{-1}$$

and hence (since $e_N \in N' \cap M_1$)

$$\lambda^{-2} = [M_1 : M] \cdot [M : N] = [M_1 : N] \geq \frac{1}{\tau_{M_1}(e_N)} + \frac{1}{1 - \tau_{M_1}(e_N)} = \frac{1}{\lambda} + \frac{1}{1 - \lambda}.$$

This yields $\lambda^{-1}(1 - \lambda) \geq 1$ and thus finally

$$[M : N] = \lambda^{-1} \geq 2.$$

An alternate, more conceptual, way of proving this is as follows. We have $(1 - e_N) \in N' \cap M_1$. Thus we can consider the subfactor

$$N(1 - e_N) \subset (1 - e_N)M_1(1 - e_N).$$

For this we have (note that $\tau_{N'}(e_N) = \tau_{M_1}(e_N)$, see proof of part (5) of Proposition 5.8)

$$[(1 - e_N)M_1(1 - e_N) : N(1 - e_N)] = [M_1 : N] \cdot \tau_{M_1}(1 - e_N) \cdot \tau_{N'}(1 - e_N) = \frac{1}{\lambda^2}(1 - \lambda)^2.$$

The index on the left hand side is ≥ 1, thus

$$\frac{1}{\lambda^2}(1 - \lambda)^2 \geq 1$$

and this implies $\lambda^{-1} \geq 2$.

q.e.d.

For further restrictions on the index we have to repeat the basic constrution. Let us start with one iteration.

Proposition 5.11.
Let $N \subset M$ be II_1 factors and $[M:N] < \infty$. Then we can repeat the basic construction to get

$$N \subset M \subset M_1 \subset M_2,$$

where

$$M_1 = \langle M, e_N \rangle \qquad \text{and} \qquad M_2 = \langle M_1, e_M \rangle.$$

Put $\lambda = [M:N]^{-1}$, then we have

$$e_M e_N e_M = \lambda e_M \qquad \text{and} \qquad e_N e_M e_N = \lambda e_N.$$

Proof. First note that (since $e_N \in M_1$)

$$e_M e_N e_M = \underbrace{E_M(e_N)}_{=\tau_{M_1}(e_N)\cdot 1 = \lambda 1} e_M = \lambda e_M.$$

Note that we cannot exchange the roles of e_N and e_M in the above proof, hence the second identity needs a different proof. For this we consider e_M, e_N as operators on $L^2(M_1)$, and we check the relation on the dense subset $\widehat{M} + \widehat{M e_N M}$. So let $x, y, z \in M$, then we have to show

$$e_N e_M e_N (x + y e_N z)\Omega = \lambda e_N (x + y e_N z)\Omega.$$

For the first summands we have

$$e_N e_M e_N x \Omega = e_N \underbrace{e_M e_N x e_M}_{E_M(e_N x) e_M} \Omega = e_N \underbrace{E_M(e_N)}_{\lambda 1} x e_M \Omega = \lambda e_N x e_M \Omega = \lambda e_N x \Omega.$$

For the second summands we have

$$\begin{aligned}
e_N e_M e_N y e_N z \Omega &= e_N \underbrace{e_M e_N y e_N z e_M}_{E_M(e_N y e_N z) e_M} \Omega \\
&= e_N E_M(\underbrace{e_N y e_N}_{E_N(y) e_N}) z e_M \Omega \\
&= e_N E_M(\underbrace{E_N(y)}_{\in N} e_N) z e_M \Omega \\
&= \underbrace{e_N E_N(y)}_{e_N y e_N} \underbrace{E_M(e_N)}_{=\lambda 1} z e_M \Omega \\
&= \lambda e_N y e_N z \Omega.
\end{aligned}$$

q.e.d.

Definition 5.12.
Let $(p_i)_{i\in I} \subset \mathcal{B}(\mathcal{H})$ be a family of projections (i.e., for each i we have $p_i^* = p_i = p_i^2$), then we denote the projection onto the sub-Hilbert space $\overline{\text{span}\,(\cup_{i\in I} p_i\mathcal{H})}$ by

$$\sup_{i\in I} p_i := \vee p_i \colon \mathcal{H} \to \overline{\text{span}\,(\cup_{i\in I} p_i\mathcal{H})}$$

and the projection onto the sub-Hilbert space $\cap_{i\in I} p_i\mathcal{H}$ by

$$\inf_{i\in I} p_i := \wedge p_i \colon \mathcal{H} \to \cap_{i\in I} p_i\mathcal{H}.$$

Remark 5.13.

1) The projections in $\mathcal{B}(\mathcal{H})$ form an *orthocomplemented lattice*, in particular we have

$$1 - p \vee q = (1-p) \wedge (1-q).$$

By using the notation $p^\perp := 1 - p$, this reads as

$$p \vee q = (p^\perp \wedge q^\perp)^\perp.$$

2) Let $M \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and $p, q \in M$ projections. Then we have that also $p \vee q \in M$ and $p \wedge q \in M$.

Proof. By the relation from (1) between $p \vee q$ and $p \wedge q$ it suffices to show the assertion for $p \wedge q$. We will do this by showing that $p \wedge q \in M'' = M$. So let $x \in M'$. We have

$$xp = px \implies \begin{cases} xp\mathcal{H} \subset p\mathcal{H} \\ x(p\mathcal{H})^\perp \subset (p\mathcal{H})^\perp \end{cases}, \qquad xq = qx \implies \begin{cases} xq\mathcal{H} \subset q\mathcal{H} \\ x(q\mathcal{H})^\perp \subset (q\mathcal{H})^\perp \end{cases}.$$

But this implies that also $(p \wedge q)\mathcal{H} = p\mathcal{H} \cap q\mathcal{H}$ and its orthogonal complement are invariant under the action of x. This implies $(p \wedge q)x = x(p \wedge q)$ for all $x \in M'$, hence $p \wedge q \in M'' = M$. q.e.d.

3) One also has

$$p \wedge q = \operatorname*{s-lim}_{n\to\infty} (pq)^n \qquad \text{(strong limit)}$$

4) Let $p, q \in \mathcal{B}(\mathcal{H})$ be projections and put

$$M := \overline{\text{alg}(p,q)}^{\text{WOT}} \subset \mathcal{B}(\mathcal{H})$$

as the weak closure of the non-unital algebra generated by p and q; the latter is the linear span of

$$p, q, pq, qp, pqp, qpq, pqpq \ldots.$$

Then M is abstractly a von Neumann algebra, with unit $p \vee q$, i.e. M leaves $(p \vee q)\mathcal{H}$ invariant and $A \subset \mathcal{B}((p \vee q)\mathcal{H})$ is a von Neumann algebra in our sense (where the identity of M must be the identity operator on the underlying Hilbert space).

Proposition 5.14.
Consider the situation as in Proposition 5.11, i.e., in the iteration M_2 of the basic construction we have two projections e_N and e_M with

$$e_M e_N e_M = \lambda e_M \qquad \text{and} \qquad e_N e_M e_N = \lambda e_N.$$

If $\lambda \neq 1$ then we have

$$e_M \vee e_N = \frac{1}{1-\lambda}(e_N + e_M - e_M e_N - e_N e_M) = \frac{1}{1-\lambda}(e_N - e_M)^2.$$

Proof. One possibility is to check that

$$1 - e_M \vee e_N = (1-e_M) \wedge (1-e_N) = \operatorname*{s-lim}_{n\to\infty} [(1-e_M)(1-e_N)]^n$$

converges to the claimed formula, by using the relations for e_N, e_M.

We prove it more directly. Put

$$x := \frac{1}{1-\lambda}(e_N + e_M - e_M e_N - e_N e_M).$$

1) Using the relations for e_N and e_M, one easily checks that x is a projection.
2) We have $x \leq e_M \vee e_N$, i.e., $x(e_M \vee e_N) = x$. This follows, since

$$e_N(e_M \vee e_N) = e_N \qquad \text{and} \qquad e_M(e_M \vee e_N) = e_M.$$

3) We have $x \geq e_M \vee e_N$. This follows from $x \geq e_N$ and $x \geq e_M$. Let us check $x \geq e_M$, i.e., $x e_M = e_M$:

$$\begin{aligned} x e_M &= \frac{1}{1-\lambda}(\cancel{e_N e_M} + e_M - \underbrace{e_M e_N e_M}_{\lambda e_M} \cancel{- e_N e_M}) \\ &= \frac{1}{1-\lambda} e_M (1-\lambda) \\ &= e_M. \end{aligned}$$

q.e.d.

Remark 5.15.
We can now use $e_M \vee e_N$ for an analogue if Proposition 5.10. But first note that for $\lambda \neq 1$

$$\tau(e_M \vee e_N) = \frac{1}{1-\lambda}(\underbrace{\tau(e_N)}_{=\lambda} + \underbrace{\tau(e_M)}_{=\lambda} - \underbrace{\tau(e_N e_M)}_{\substack{=\tau(e_M)\tau(e_N)\\ =\lambda^2}} - \underbrace{\tau(e_M e_N)}_{\lambda^2}) = \frac{1}{1-\lambda}(2\lambda - 2\lambda^2) = 2\lambda.$$

For any projection p we have $\tau(p) \leq 1$, hence we have $2\lambda \leq 1$, i.e.

$$[M : N] = \frac{1}{\lambda} \geq 2.$$

This gives another proof for Proposition 5.10 that the interval $(\frac{1}{2}, 1)$ is forbidden for $\lambda = [M : N]^{-1}$.

Let us see whether we can exclude more for $\lambda \leq 1/2$. For this consider

$$N \subset M \subset M_1 \subset M_2$$

and note that $e_N, e_M \in N' \cap M_2$, hence also $e_N \vee e_M \in N' \cap M_2$, and

$$[M_2 : N] = [M_2 : M_1] \cdot [M_1 : M] \cdot [M : N] = \frac{1}{\lambda^3}.$$

Hence

$$\begin{aligned}
&[(1 - e_N \vee e_M)M_2(1 - e_N \vee e_M) : N(1 - e_N \vee e_M)] \\
&\qquad\qquad = [M_2 : N] \cdot \underbrace{\tau_{M_2}(1 - e_N \vee e_M)}_{=1-2\lambda} \cdot \underbrace{\tau_{N'}(1 - e_N \vee e_M)}_{=\tau_{M_2}(\dots)=1-2\lambda} \\
&\qquad\qquad = \frac{1}{\lambda^3}(1 - 2\lambda)^2.
\end{aligned}$$

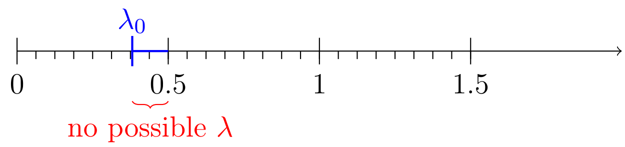
Since the index is ≥ 1, this implies

$$\frac{1}{\lambda^3}(1 - 2\lambda)^2 \geq 1 \qquad \text{thus} \qquad (\lambda - 1)(\lambda^2 - 3\lambda + 1) \leq 0.$$

Since $\lambda < 1$ we get $\lambda^2 - 3\lambda + 1 \geq 0$. Hence values of λ for which we have $\lambda^2 - 3\lambda + 1 < 0$ are forbidden. By elementary observations we find this is the case when $\lambda \in (\lambda_0, \frac{1}{2})$, where

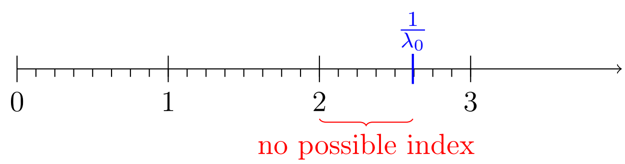
$$\lambda_0 = \frac{3 - \sqrt{5}}{2}.$$

This leads the following picture.



Hence there is no index in $(2, \frac{1}{\lambda_0})$, where

$$\frac{1}{\lambda_0} = 4\cos^2\left(\frac{\pi}{5}\right) \approx 2.618.$$



Note that in the argument above we needed $1 - e_N \vee e_M \neq 0$, i.e., $\lambda = \frac{1}{2}$ is not excluded because in this case $\tau(e_N \vee e_M) = 1$ and hence $e_N \vee e_M = 1$. Thus an index 2 is still possible.

Up to now we have seen that in the interval $[1, \frac{1}{\lambda_0}]$ the only possible values for indices are

$$1 = 4\cos^2(\frac{\pi}{3}), \quad 2 = 4\cos^2(\frac{\pi}{4}), \quad \frac{1}{\lambda_0} = 4\cos^2(\frac{\pi}{5}).$$

Further restrictions come from further iterations of the basic construction. In the next section this will be systematized.

6 Jones Tower and Proof of Jones Theorem

Definition 6.1.
Let $N \subset M$ be II_1 factors of finite index, $[M : N] < \infty$. Put

$$M_{-1} := N, \quad M_0 := M$$

and define inductively

$$M_{i+1} = \langle M_i, e_{M_{i-1}} \rangle = \langle M_i, e_{i+1} \rangle,$$

where we put $e_i := e_{M_{i-2}}$. We call this the *Jones tower of factors.* The sequence of the projections $(e_i)_{i\in\mathbb{N}}$ is called the sequence of the *Jones projections.*

We will also write the above as

$$N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} \dots \overset{e_{i-1}}{\subset} M_{i-1} \overset{e_i}{\subset} M_i \overset{e_{i+1}}{\subset} \dots$$

Proposition 6.2.
The e_i enjoy the following properties; as usually, we put $\lambda = [M : N]^{-1}$.

1) $e_i^2 = e_i = e_i^*$,
2) $e_i e_j = e_j e_i$, if $|i - j| \geq 2$,
3) $e_i e_{i\pm1} e_i = \lambda e_i$,
4) $\tau(w e_{n+1}) = \lambda \tau(w)$, for any word w on $\{e_1, \dots, e_n\}$.

Proof.

1) The e_i are defined as the projections onto M_{i-1}.
2) By Proposition 5.4 we have $e_i \in M'_{i-2}$, and $e_{i-k} \in M_{i-k} \subset M_{i-2}$ for $k \geq 2$.
3) This follows from Proposition 5.11.
4) This follows by part (6) of Proposition 5.8.

q.e.d.

Notation 6.3.

1) For a sequence of Jones projections $(e_i)_{i\in\mathbb{N}}$ we define the *Jones-Wenzl projections* $(f_n)_{n\in\mathbb{N}}$ by

$$f_n = 1 - e_1 \vee \dots \vee e_n = (1 - e_1) \wedge \dots \wedge (1 - e_n)$$

2) We define polynomials $(P_n)_{n\geq0}$ by the following recursion. We put

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 \\ P_{n+1}(x) &= P_n - xP_{n-1} \qquad (n \geq 1) \end{aligned}$$

Remark 6.4.

1) We have $P_0(x) = 1$, $P_1(x) = 1$, and

$$P_2(x) = 1 - x, \quad P_3(x) = 1 - 2x, \quad P_4(x) = 1 - 3x + x^2, \quad \dots$$

2) Our main goal is to see that

$$\tau(f_n) = P_{n+1}(\lambda) \qquad \text{if } P_k(\lambda) \neq 0 \text{ for } k = 1, \dots, n+1.$$

Since some $P_{n+1}(\lambda)$ will become negative and traces of projections can never be negative, we need that $P_k(\lambda) = 0$ for some k. This will then give restrictions on $[M : N] = \lambda^{-1}$.

3) For small m we have

$$\begin{aligned}
\tau(f_0) &= \tau(1) = 1 = P_1(\lambda) \\
\tau(f_1) &= \tau(1 - e_1) = 1 - \lambda = P_2(\lambda) \\
\tau(f_2) &= \tau(1 - e_1 \vee e_2) = 1 - 2\lambda = P_3(\lambda) \\
\tau(f_3) &= \tau(1 - e_1 \vee e_2 \vee e_3) = 1 - 3\lambda + \lambda^2 = P_4(\lambda)
\end{aligned}$$

4) The P_n are just the Chebyshev polynomials S_n in disguise. Define

$$S_n := x^n P_n\left(\frac{1}{x^2}\right)$$

then

$$\begin{aligned}
S_0(x) &= P_0\left(1/x^2\right) = 1 \\
S_1(x) &= xP_1\left(1/x^2\right) = x \\
S_2(x) &= x^2 P_2\left(1/x^2\right) = x^2\left(1 - \frac{1}{x^2}\right) = x^2 - 1 \\
S_3(x) &= x^3 P_3\left(1/x^2\right) = x^3\left(1 - 2\frac{1}{x^2}\right) = x^3 - 2x
\end{aligned}$$

and

$$\begin{aligned}
xS_n(x) &= x^{n+1} P_n\left(1/x^2\right) \\
&= x^{n+1}\left[P_{n+1}\left(1/x^2\right) + \frac{1}{x^2} P_{n-1}\left(1/x^2\right)\right] \\
&= S_{n+1}(x) + S_{n-1}(x).
\end{aligned}$$

This defines the Chebyshev polynomials of the second kind. They have the following properties.

a) The polynomials $(S_n)_{n\geq 0}$ are orthonormal w.r.t. the semicircular distribution on $[-2,2]$, i.e.

$$\int_{-2}^{2} S_n(x)S_m(x)\frac{1}{2\pi}\sqrt{4-x^2}dx = \delta_{nm}.$$

b) We have the "explicit" formula

$$S_n(2\cos\vartheta) = \frac{\sin(n+1)\vartheta}{\sin\vartheta}.$$

Hence $S_n(x) = 0$ for $x = 2\cos\vartheta$ with $\sin(n+1)\vartheta = 0$ and $\theta \neq 0$, i.e., for

$$\vartheta = \frac{\pi}{n+1}k \qquad (k = 1,\dots,n).$$

The corresponding zeros of

$$P_n\left(\frac{1}{x^2}\right) = \frac{1}{x^n}S_n(x)$$

are then at

$$\frac{1}{x^2} = \frac{1}{4\cos^2\left(\frac{\pi}{n+1}k\right)}.$$

Note that k is only running over $1,\dots,[\frac{n}{2}]$. For n odd, the case $k = (n+1)/2$ gives $\vartheta = \pi/2$, hence does not show up; because of the square, $\vartheta = \pi/2 + y$ and $\vartheta = \pi/2 - y$ give the same zero of P_n.

5) Let us now come back to the Jones-Wenzl projections

$$f_n = 1 - e_1 \vee \cdots \vee e_n = (1-e_1) \wedge \cdots \wedge (1-e_n).$$

Note that f_n is the largest projection with the property that

$$f_n e_i = 0 \quad (\text{i.e. } f_n \perp e_i) \qquad \text{for all } i = 1,\dots,n.$$

Proposition 6.5.
Assume that $P_i(\lambda) \neq 0$ for $i = 1,\dots,n$. Then we have

$$f_n = f_{n-1} - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_n f_{n-1}.$$

Proof. We prove the assertion by induction. For $n = 2$ we have

$$\begin{aligned} f_1 - \frac{P_1(\lambda)}{P_2(\lambda)} f_1 e_2 f_1 &= 1 - e_1 - \frac{1}{1-\lambda}(1-e_1)e_2(1-e_1) \\ &= 1 - e_1 - \frac{1}{1-\lambda}(e_2 - e_2e_1 - e_1e_2 + \underbrace{e_1e_2e_1}_{\lambda e_1}) \\ &= 1 - \frac{1}{1-\lambda}(e_1 + e_2 - e_2e_1 - e_1e_2) \\ &= 1 - e_1 \vee e_2 \\ &= f_2 \end{aligned}$$

In the last step we used Proposition 5.14. Note that for this we need $\lambda \neq 1$; this is given because of our assuption that $1 - \lambda = P_2(\lambda) \neq 0$.

Now we assume that our assertion holds for some $n-1 \in \mathbb{N}$, i.e., we have

$$f_{n-1} = f_{n-2} - \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} f_{n-2} e_{n-1} f_{n-2}.$$

Put

$$x = f_{n-1} - \frac{P_{n-1}(\lambda)}{P_n \lambda)} f_{n-1} e_n f_{n-1}.$$

We have to show that $x = f_n$, i.e.,

1) x is a projection.

2) $x \leq f_n$, i.e. $f_n \perp e_i$ for all $i = 1, \ldots, n$.

3) $x \geq f_n$.

First we calculate

$$\begin{aligned} e_n f_{n-1} e_n &= e_n \left[f_{n-2} - \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} f_{n-2} e_{n-1} f_{n-2} \right] e_n \\ &= e_n f_{n-2} e_n - \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} e_n f_{n-2} e_{n-1} f_{n-2} e_n \end{aligned}$$

Since $e_k e_n = e_n e_k$ for $1 \leq k \leq n-2$ we have $f_{n-2} e_n = e_n f_{n-2}$. Thus we can continue as follows.

$$\begin{aligned} e_n f_{n-1} e_n &= e_n f_{n-2} - \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} f_{n-2} \underbrace{e_n e_{n-1} e_n}_{=\lambda e_n} f_{n-2} \\ &= e_n f_{n-2} - \lambda \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} \underbrace{f_{n-2} e_n f_{n-2}}_{=e_n f_{n-2}} \\ &= e_n f_{n-2} \left(1 - \lambda \frac{P_{n-2}(\lambda)}{P_{n-1}(\lambda)} \right) \\ &= e_n f_{n-2} \left(\frac{P_{n-1}(\lambda) - \lambda P_{n-2}(\lambda)}{P_{n-1}(\lambda)} \right) \\ &= e_n f_{n-2} \frac{P_n(\lambda)}{P_{n-1}(\lambda)}. \end{aligned}$$

1) It is obvious that $x = x^*$; furthermore we have (note that $f_{n-1} \leq f_{n-2}$)

$$\begin{aligned}
x^2 &= f_{n-1} - 2\frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_nf_{n-1} + \frac{P_{n-1}^2(\lambda)}{P_n^2(\lambda)} f_{n-1} \underbrace{e_nf_{n-1}e_n}_{e_nf_{n-2}\frac{P_n(\lambda)}{P_{n-1}(\lambda)}} f_{n-1} \\
&= f_{n-1} - 2\frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_nf_{n-1} + \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_n \underbrace{f_{n-2}f_{n-1}}_{=f_{n-1}} \\
&= f_{n-1} - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_nf_{n-1} \\
&= x.
\end{aligned}$$

2) The fact that $xe_i = 0$ for $i = 1, \dots, n-1$ is immediate since $f_{n-1}e_i = 0$ for all $i = 1, \dots, n-1$ (see Remark 6.4(5)). Thus it remains to show that $xe_n = 0$. We will show instead $e_nxe_n = 0$; this implies the former via

$$0 = e_nxe_n = e_nxxe_n = (xe_n)^*(xe_n) \implies xe_n = 0.$$

We have

$$\begin{aligned}
e_nxe_n &= e_nf_{n-1}e_n - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} e_nf_{n-1}e_nf_{n-1}e_n \\
&= e_nf_{n-1}e_n - e_nf_{n-1} \underbrace{e_nf_{n-2}}_{=f_{n-2}e_n} \\
&= e_nf_{n-1}e_n - e_n \underbrace{f_{n-1}f_{n-2}}_{f_{n-1}} e_n \\
&= 0.
\end{aligned}$$

3) We have (note that $e_n \perp f_n$)

$$\begin{aligned}
xf_n &= f_{n-1}f_n - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1}e_n \underbrace{f_{n-1}f_n}_{f_n} \\
&= f_n - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1} \underbrace{e_nf_n}_{=0} \\
&= f_n,
\end{aligned}$$

This shows that $f_n \leq x$.

q.e.d.

Proposition 6.6.
Assume that $P_i(\lambda) \neq 0$ for $i = 1, \dots, n$. Then we have

$$\tau(f_n) = P_{n+1}(\lambda).$$

Proof. By Proposition 6.5 and the fact that

$$\tau(f_{n-1}e_nf_{n-1}) = \tau(f_{n-1}) \cdot \tau(e_n) = \lambda\tau(f_{n-1}),$$

we have

$$\begin{aligned}
\tau(f_n) =& \tau(f_{n-1}) - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} \underbrace{\tau(f_{n-1}e_nf_{n-1})}_{=\lambda\tau(f_{n-1})} \\
=& \tau(f_{n-1}) \left(\frac{P_n(\lambda) - \lambda P_{n-1}(\lambda)}{P_n(\lambda)} \right) \\
=& \tau(f_{n-1}) \frac{P_{n+1}(\lambda)}{P_n(\lambda)} \\
=& \tau(f_{n-2}) \frac{P_n(\lambda)}{P_{n-1}(\lambda)} \frac{P_{n+1}(\lambda)}{P_n(\lambda)} \\
=& \tau(f_{n-2}) \frac{P_{n+1}(\lambda)}{P_{n-1}(\lambda)} \\
& \vdots \\
=& \tau(f_1) \frac{P_{n+1}(\lambda)}{P_1(\lambda)} \\
=& (1-\lambda) \frac{P_{n+1}(\lambda)}{1-\lambda} \\
=& P_{n+1}(\lambda).
\end{aligned}$$

q.e.d.

Theorem 6.7 (Jones)**.**
Let $N \subset M$ be II_1 factors. Then if $[M : N] < 4$ we have

$$[M : N] = 4\cos^2\left(\frac{\pi}{n}\right) \qquad \text{for some } n = 3, 4, 5 \dots$$

Proof. Assume we have for $\lambda \in (\frac{1}{4}, 1]$ that $P_i(\lambda) \neq 0$ for $i = 1, \dots, n$ and $P_{n+1}(\lambda) < 0$. Then such a λ cannot show up as $\lambda = [M : N]^{-1}$, because then one would have

$$0 \leq \tau(f_n) = P_{n+1}(\lambda) < 0.$$

Denote by I_n the open interval

$$I_n := \left(\frac{1}{4\cos^2\left(\frac{\pi}{n+2}\right)}, \frac{1}{4\cos^2\left(\frac{\pi}{n+1}\right)} \right).$$

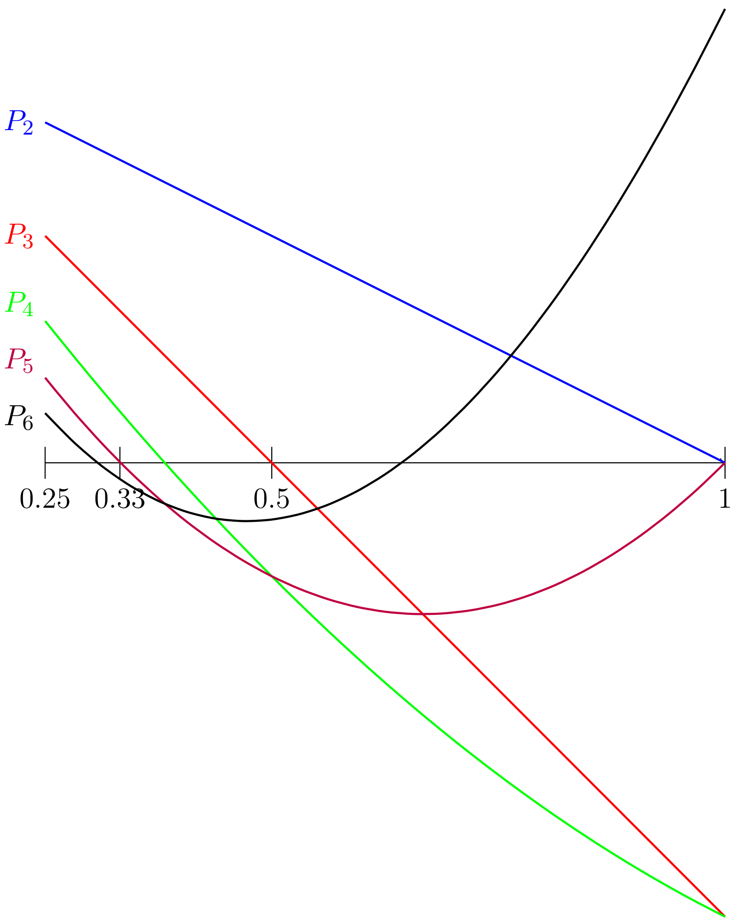
Then it is easy to check that we have for $\lambda \in I_n$:

$$P_i(\lambda) > 0 \quad \text{for } i = 1, \dots, n \qquad \text{and} \qquad P_{n+1}(\lambda) < 0.$$

(See below for a plot of some of the P_n.) Hence numbers in the intervals

$$(4\cos^2(\pi/(n+1)), 4\cos^2(\pi/(n+2)))$$

are forbidden as index. For the endpoints of those intervals our argument does not work and those might appear as index. q.e.d.



7 Realization of Index Less Than 4 and Temperley-Lieb Algebras

Remark 7.1.

1) We know that the possible index-values are
   a) $[M : N] \geq 4$
   b) $[M : N] = 4\cos^2\left(\frac{\pi}{n}\right)$ for $n = 3, 4, 5, \ldots$

   Moreover we know (see Example 4.8) that in the case a) we always have a subfactor but for b) we have no result so far.

2) It was shown by Jones, that all values in (b) can actually occur. The idea for this is the following: If we find a sequence of projections $e_1, e_2 \ldots$ somewhere in some finite von Neumann algebra M which satisfy the properties of the Jones projections from Proposition 6.2, then we define

$$P = \{e_1, e_2, \ldots\}'' \subset M, \qquad Q = \{e_2, e_3 \ldots\}'' \subset M.$$

   Then one has to show that $Q \subset P$ are II_1 factors and that $[P : Q] = 1/\lambda$.

3) Finitely many of the projections, $e_1, \ldots, e_n$, can be realized in finite-dimensional setting, then we can take the inductive limit (and GNS construction w.r.t. trace τ) of those.

4) The first step we will do is to give up the requirement, that $e_i^* = e_i$.

Definition 7.2.
For $\lambda \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$, the *Temperley-Lieb algebra* $TL_{n+1}(\lambda)$ is the unital algebra with generators $e_1, \ldots, e_n$ and the relations

1) $e_i^2 = e_i$,
2) $e_i e_{i\pm1} e_i = \lambda e_i$,
3) $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

Remark 7.3.

1) The algebras $TL_n(\lambda)$ are finite dimensional. The relations allow us to bring any monomial in the generators e_i in a reduced form

$$(e_{i_1} e_{i_1-1} \ldots e_{k_1})(e_{i_2} e_{i_2-1} \ldots e_{k_2}) \ldots (e_{i_p} e_{i_p-1} \ldots e_{k_p})$$

   with strictly increasing indices $i_1 < i_2 < \cdots < i_p$ and strictly increasing indices $k_1 < k_2 < \cdots < k_p$. An example of such a reduced form is

$$(e_3 e_2 e_1)(e_4 e_3)(e_5 e_4) \in TL_5.$$

The number of such reduced terms are counted by the Catalan numbers

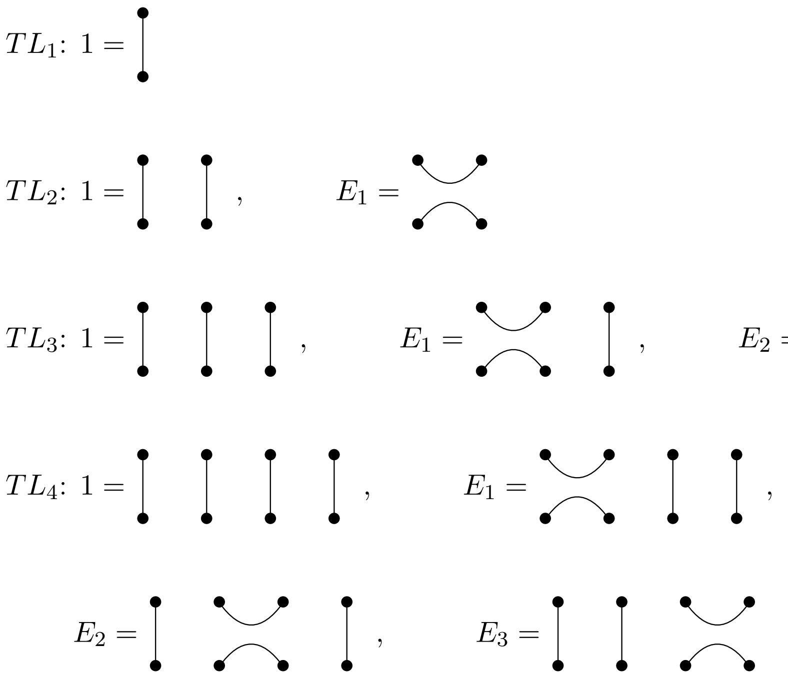
$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Here are these reduced terms for the first n:

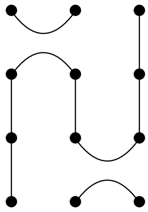
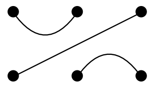
TL_1	1	$c_1 = 1$
TL_2	$1, e_1$	$c_2 = 2$
TL_3	$1, e_1, e_2, e_2e_1, e_1e_2$	$c_3 = 5$
TL_4	$1, e_1, e_1e_2, e_1e_3, e_1e_3e_2, e_1e_2e_3, e_3e_2e_1,$ $e_2, e_2e_1, e_2e_3, e_2e_1e_3, e_2e_1e_3e_2, e_3, e_3e_2$	$c_4 = 14$

Hence we have that $\dim TL_n \leq c_n$.

2) We have a diagrammatic representation of TL_n, via strings connecting $2n$ points, n on a top line and n on a bottom line.

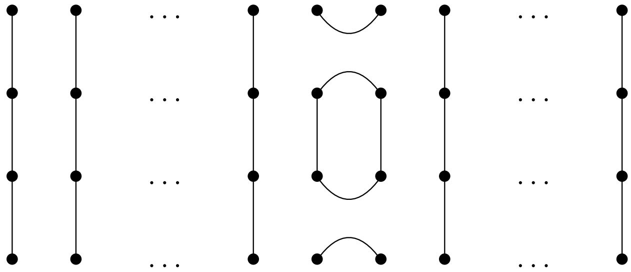


The multiplication is given by putting the pictures one upon the other, connecting and then erasing the middle points and getting a factor α for each loop. Here is an example.

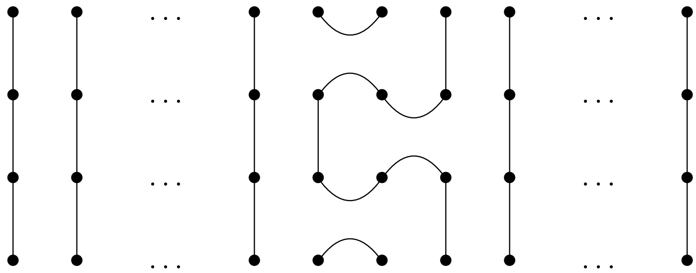
$E_1 E_2 =$  $=$ 

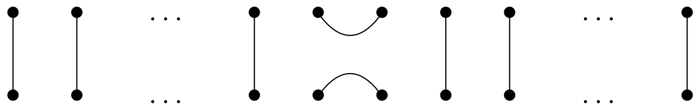
Hence we have the relations.

- The graph corresponding to the 1 is indeed a unit.
- We have

  $E_i^2 =$  $= \quad \alpha E_i$

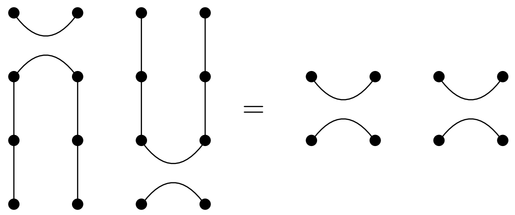
  where α is the weight of a closed loop.
- We have

  $E_i E_{i+1} E_i =$ 

  $=$ 

  $= E_i$

- We have

$E_1E_3 =$  $= E_3E_1$

Thus the $e_i = \frac{1}{\alpha}E_i$ satisfy

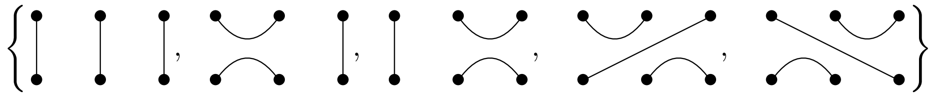
a)

$$e_i^2 = \frac{1}{\alpha^2}E_i^2 = \frac{1}{\alpha}E_i = e_i,$$

b)

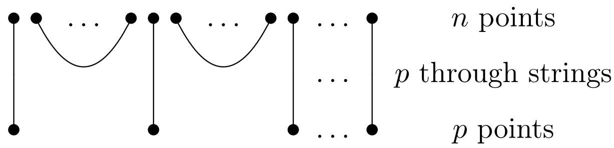
$$e_i e_{i\pm1} e_i = \frac{1}{\alpha^3}E_iE_{i\pm1}E_i = \frac{1}{\alpha^3}E_i = \frac{1}{\alpha^2}e_i.$$

Hence for $\lambda = \frac{1}{\alpha^2}$ the e_i satisfy the Temperley-Lieb relations.
Note that the $E_1, \dots, E_n$ generate $NC(2n)$, the non-crossing pairings of $2n$ points. For instance, $NC(6)$ is given by $NC(6) = \{1, E_1, E_2, E_1E_2, E_2E_1\}$, i.e.,



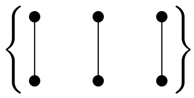
Since there are c_n many non-crossing pairings of $2n$ points, and those diagrams are linearly independent, we have for all n and all λ that $\dim(TL_n(\lambda)) = c_n$.

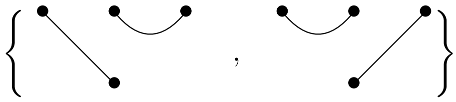
3) What is the structure of the algebra TL_n? We can also give repesentations in diagrammatic terms (by acting with the algebra on itself and then decomposing). We denote by $V_{n,p}$ the vector space generated by pictures of the form



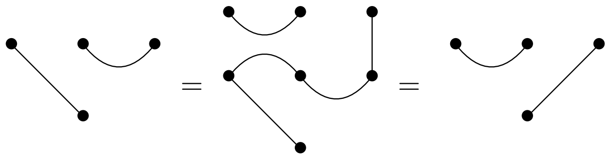
We have here n points on the upper level and $p \leq n$ points on the lower level. The upper points can be paired among themselves (in a non-crossing way), but the lower points must all be paired to upper points. Hence there will be p through strings (strings which connect an upper with a lower point). Diagrams from TL_n act on diagrams from $V_{n,p}$ by concatenation and removing loops (counted by a factor α). If the number of through-strings decreases by doing so, then the result is zero. Note that it is not possible to increase the number of through strings. If our parameter λ is "generic", then the TL_n are semi-simple and the spaces $V_{n,p}$ give irreducible representations. In the following we will assume that we are in such generic cases. (This is for example the case, for $\lambda \leq 1/4$.)
Let us consider some examples.

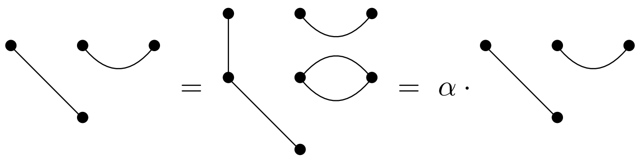
a) For $n = 3$ we can have $p = 3$ or $p = 1$, giving the two vector spaces:

$V_{3,3} = \text{span}$ 

$V_{3,1} = \text{span}$ 

Here is the action of E_1 and E_2 on the first element from $V_{3,1}$

$E_1 \cdot$ 

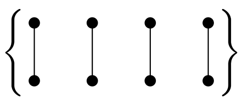
$E_2 \cdot$ 

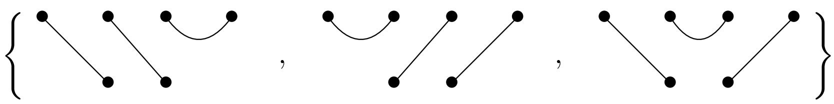
TL_3 acts now on $V_{3,1}$ as $M_1(\mathbb{C})$ and on $V_{3,3}$ as $M_2(\mathbb{C})$, hence we have

$$TL_3 = M_1(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Note that this gives the right dimension $1^2 + 2^2 = 5 = c_2$.

b) For $n = 4$, we can have $p = 4$, $p = 2$, or $p = 0$, giving the vector spaces

$V_{4,4} = \text{span}$ 

$V_{4,2} = \text{span}$ 

$V_{4,0} = \text{span}$ 

In the generic situation TL_4 acts on $V_{4,4}$ as M_1, on $V_{4,2}$ as M_3 and on $V_{4,0}$ as M_2. Thus we have

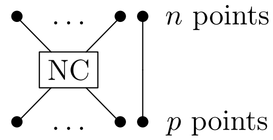
$$TL_4 \cong M_1 \oplus M_3 \oplus M_2$$

which gives again the right dimension $1^2 + 3^2 + 2^2 = 14 = c_4$.

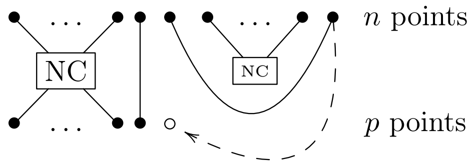
In general we have

$$TL_n = \bigoplus_{r=0}^{[\frac{n}{2}]} \mathcal{B}(V_{n,n-2r}).$$

4) Note that in $V_{p,n}$ we two possibilities for a diagram. The last upper point can be connected either to the last lower point ....



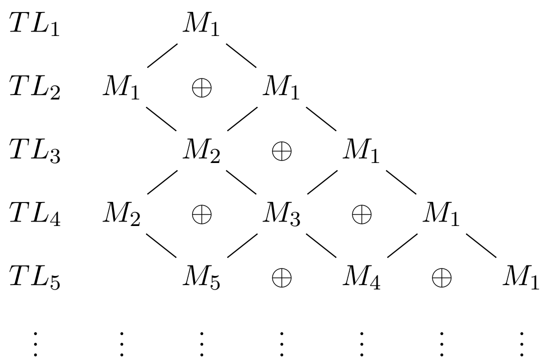
... or to one of the upper points ...



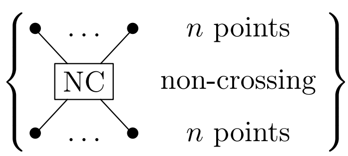
In the first case such a diagram can be identified with a diagram in $V_{n-1,p-1}$ by removing the last points on the upper and lower line; in the second case such a diagram can be identified with a diagram in $V_{n-1,p+1}$ by moving the last upper point to the lower line. This identification gives a bijection, thus we have

$$V_{n,p} \cong V_{n-1,p-1} \oplus V_{n-1,p+1}.$$

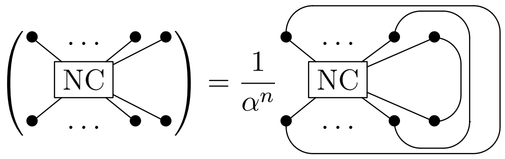
Since this identification intertwines with the action of TL_n, we have for the inclusion of the TL_ns the following *Bratteli diagram*.



Definition 7.4.
On the linear span of the $NC(2n)$ diagrams

$TL_n(\lambda) = \text{span}$ 

we define a linear functional $\tau_n \colon TL_n(\lambda) \to \mathbb{C}$ by linear extension of

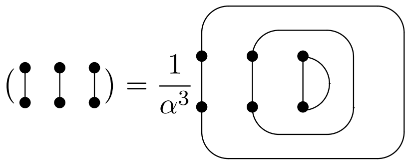
τ_n  $= \frac{\alpha^{\#\text{loops}}}{\alpha^n}$,

where we have put

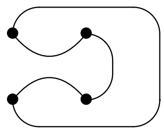
$$\frac{1}{\alpha^2} = \lambda.$$

Example 7.5.
Let us calculate a few examples.

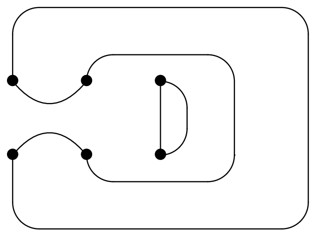
1) τ_n is unital:

$\tau_3(1) = \tau_3$  $= \frac{1}{\alpha^3}\alpha^3 = 1.$

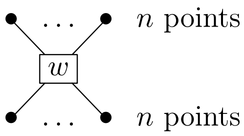
2) On the E_i we get α as value. Consider as example $E_1 \in TL_2$:

$\tau_2(E_1) = \frac{1}{\alpha^2}$  $= \frac{1}{\alpha^2}\alpha = \alpha.$

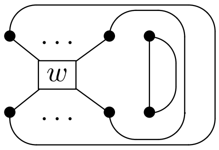
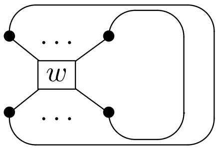
3) Note that τ is compatible with the embeddings of $TL_n \subset TL_m$ for $n < m$. For example, let us redo the calculation from (2), but now in TL_3:

$\tau_3(E_1) = \frac{1}{\alpha^3}$  $= \frac{1}{\alpha^3}\alpha^2 = \alpha = \tau_2(E_1).$

More generally consider a diagram $w \in TL_n$, which we write in the form

$w =$  .

The calculation of τ_{n+1} in TL_{n+1} gives then

$$\tau_{n+1}(w) = \frac{1}{\alpha^{n+1}}$$

$$= \frac{1}{\alpha^{n}}$$

$$= \tau(w).$$

This compatibility under the embedding goes over from diagrams to their span, hence $\tau_{n+1}\big|_{TL_n} = \tau_n$, and we can define τ consistently on all TL_n,

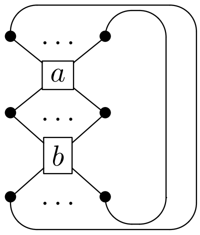
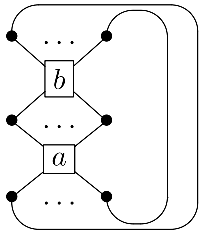
$$\tau\colon \bigcup_{n=1}^{\infty} TL_n(\lambda) \to \mathbb{C}.$$

Proposition 7.6.
The map τ defined defined above has the following properties.

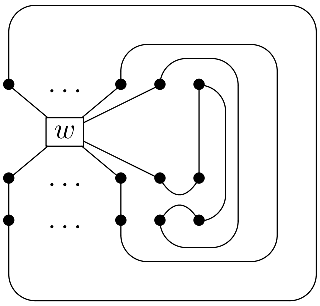
1) $\tau(1) = 1$,
2) $\tau(ab) = \tau(ba)$ for all $a, b \in \bigcup_{n=1}^{\infty} TL_n(\lambda)$,
3) $\tau(we_n) = \tau(w)\lambda$ for all $w \in TL_{n-1}$ and all $n \in \mathbb{N}$.

Proof.

1) This is immediately clear.
2) It suffices to check it for diagrams a and b; by linearity it follows then also for linear combinations. We can embedd a, b in the same TL_n, then we have

$$\tau(ab) \quad =$$

$$=$$

$$= \quad \tau(ba)$$

3) Consider a diagram $w \in TL_n(\lambda)$, then we have

$$\tau(we_n) \quad = \quad \frac{1}{\alpha}\tau_{n+1}(wE_n) \quad = \quad \frac{1}{\alpha^{n+2}}$$

$$= \quad \frac{1}{\alpha^2}\tau_n(w) \quad = \quad \lambda\tau(w)$$

The statement for general elements of $TL_{n-1}(\lambda)$ follows again by linearity.

q.e.d.

Remark 7.7.
We ask how the trace τ does look like in the representation

$$TL_n = \bigoplus_{r=0}^{\left[\frac{n}{2}\right]} \mathcal{B}(V_{n,n-2r}).$$

It must be of the form

$$\tau_n = \bigoplus_r \gamma_{n,r} \operatorname{Tr}_{(r)},$$

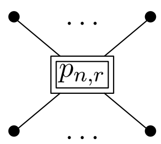
where $\operatorname{Tr}_{(r)}$ is the unnormalized trace on $\mathcal{B}(V_{n,n-2r})$. We will now try to determine the coefficients $\gamma_{n,r}$. If we choose a minimal projection

$$p_{n,r} \in \mathcal{B}(V_{n,n-2r}) \subset \bigoplus_k \mathcal{B}(V_{n.n-2k}) \cong TL_n,$$

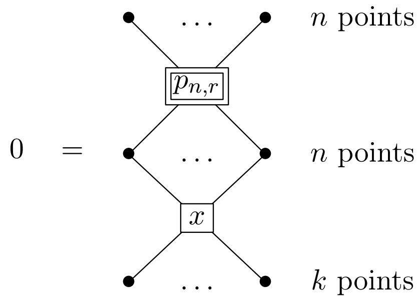
then we have

$$\gamma_{n,r} = \tau_n(p_{n,r}).$$

Note that this $p_{n,r}$ acts on $\bigoplus_k \mathcal{B}(V_{n.n-2k})$ via $(\dots,\xi,\dots) \mapsto (0,\dots,0,P_{n,r}\xi,0,\dots,0)$. This means that $p_{n,r}$ corresponds in TL_n to a linear combination of diagrams, with the property of acting as zero on all $V_{n,n-2l}$ for $l \neq r$. If we present the linear combinations of diagrams for $p_{n,r}$ as



then we have for a diagram $x \in V_{n,k}$



if $k \neq n - 2r$.

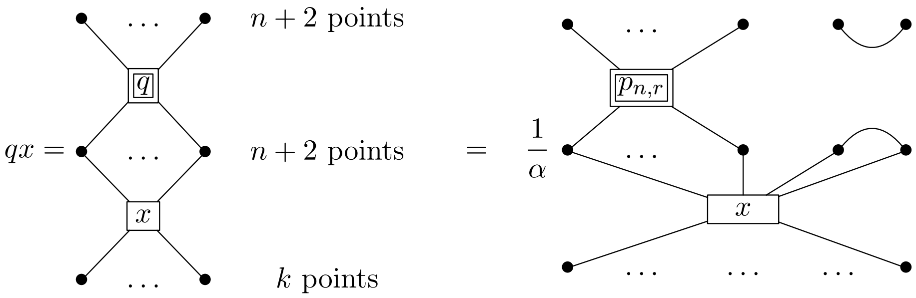
Lemma 7.8.
$p_{n,r}e_{n+1}$ is a minimal projection on $V_{n+2,n+2-2(r+1)}$ and we have

$$\gamma_{n+2,r+1} = \lambda\gamma_{n,r}.$$

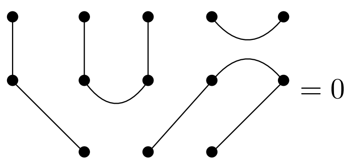
Proof. The projection $p_{n,r}$ contains only $e_1, \dots, e_{n-1}$, hence $p_{n,r}$ and e_{n+1} commute and thus $q := p_{n,r}e_{n+1}$ is a projection. We have to show that $qx = 0$ for all $x \in V_{n+2,k}$ with $k \neq n+2-2(r+1)$. If this is proven then $q \in \mathcal{B}(V_{n+2,n+2-2(r+1)})$ is a minimal projection and we have

$$\gamma_{n+2,r+1} = \tau(q) = \tau(p_{n,r}e_{n+1}) = \tau(p_{n,r})\underbrace{\tau(e_{n+1})}_{\lambda} = \tau(p_{n,r})\lambda = \lambda\gamma_{n,r}.$$

So it remains to prove that $qx = 0$ for all $x \in V_{n+2,k}$ with $k \neq n+2-2(r+1)$. Consider such an $x \in V_{n+2,k}$. Then



The closed string on the upper right hand side of x can remove some of the k through strings – like in the example

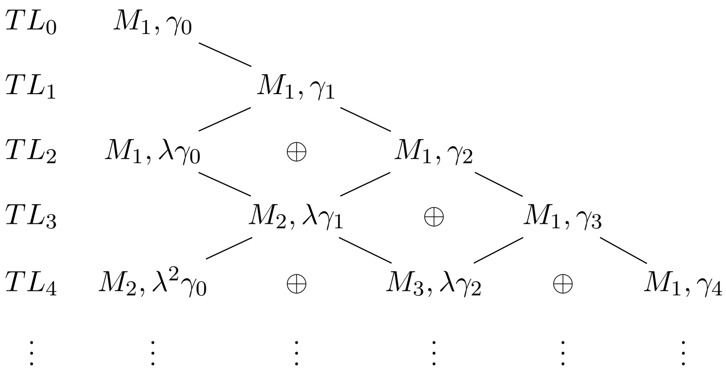


– but then the action is zero. If no through strings are removed, then we must have

$$k = n - 2r = n + 2 - 2(r+1);$$

hence x must be in $V_{n+2,n+2-2(r+1)}$, otherwise $qx = 0$. q.e.d.

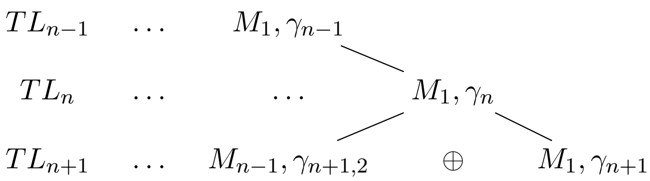
Remark 7.9.
The recursion $\gamma_{n+2,r+1} = \lambda\gamma_{n,r}$ determines all $\gamma_{n,r}$ in terms of $\gamma_{n,0} =: \gamma_n$, according to the following pattern



Since our trace is unital, we have some more relations on each row which allows to calculate also the values of the γ_n. Let us do this explicitly for small n. Clearly we must have $\gamma_0 = 1$ and $\gamma_1 = 1$. For $n = 2, 3, 4$ we get then

$$\begin{array}{lcll} 1 = \tau_2(1) = \lambda + \gamma_2 & \Longrightarrow & \gamma_2 = 1 - \lambda & = P_2(\lambda) \\ 1 = \tau_3(1) = 2\lambda + \gamma_3 = 1 & \Longrightarrow & \gamma_3 = 1 - 2\lambda & = P_3(\lambda) \\ 1 = \tau_4(1) = 2\lambda\gamma_1 + 3\lambda\gamma_2 + \gamma_4 = 1 & \Longrightarrow & \gamma_4 = 1 - 3\lambda + \lambda^2 & = P_4(\lambda) \end{array}$$

To see that we have indeed in general the equality between the γ_n and our disguised Chebyshev polynomials P_n we have to consider the following part of our Bratteli diagram.



Thus the image of the minimal projection $p_{n,0}$ splits into two minimal projections on the level $n+1$, hence

$$\gamma_n = \underbrace{\gamma_{n+1,2}}_{=\lambda\gamma_{n-1}} + \gamma_{n+1},$$

such that $\gamma_{n+1} = \gamma_n - \lambda\gamma_{n-1}$, which is the recursion for the polynomials P_n. Hence we have shown the following theorem.

Theorem 7.10.
Our traces on $TL_n \cong \bigoplus_r \mathcal{B}(V_{n,n-2r})$ are given by

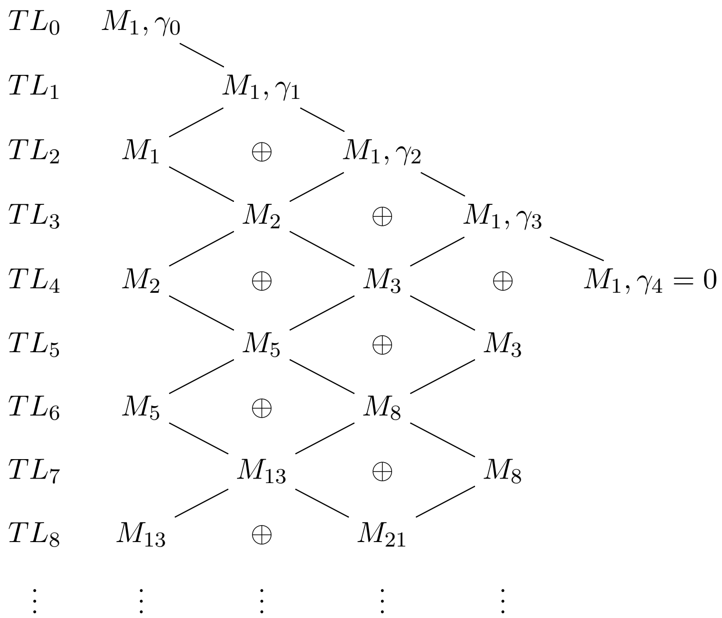
$$\tau_n = \bigoplus_r \gamma_{n,r} \operatorname{Tr}_{(r)} \qquad \text{with} \qquad \gamma_{n,r} = \lambda^r P_{n-2r}(\lambda),$$

where $\operatorname{Tr}_{(r)}$ is the unnormalized trace on $\mathcal{B}(V_{n,n-2r})$.

Remark 7.11.

1) If $0 \leq \lambda \leq \frac{1}{4}$ (i.e., $[M : N] \geq 4$), then $P_k(\lambda) \geq 0$ for all k and consequently $\gamma_{n,r} \geq 0$ for all appropriate n, r. Hence we can realize arbitrarily many $e_1, \ldots e_n$ with respect to a positive trace.
2) If $\lambda > \frac{1}{4}$ then some $P_k(\lambda)$ will become negative. In order to keep the trace positive, we must hit zero somewhere and then truncate the Bratteli diagram from this point on (and adjust the weights for the trace).

   As an example, assume that λ is the smallest zero of P_4; thus $P_1(\lambda), P_2(\lambda), P_3(\lambda) > 0$ and $P_4(\lambda) = 0$; then we get the truncated Bratteli diagram



For the values of λ of the form $(4\cos^2(\pi/n))^{-1}$ we know that we hit zero before getting negative values, hence those yield such valid truncated Bratteli diagrams. If we then take inductive limits of the truncated diagrams and do the GNS construction with respect to the trace, then this gives a realization of subfactors $N \subset M$ with index of the form

$$[M : N] = \frac{1}{\lambda} = 4\cos^2\left(\frac{\pi}{n}\right) < 4 \qquad (n \geq 3).$$

Of course, there are many things to check. For example, it is not clear why we should get factors via this GNS-construction. We will come back to this question.

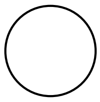
8 Braids, Knots and the Jones Polynomials

Definition 8.1.

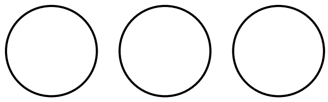
1) A *knot* K is a faithful, smooth embedding $\varphi\colon S^1 \to \mathbb{R}^3$.
2) A *link* L is a faithful, smooth embedding $\varphi\colon \bigcup_{\text{finite}} S^1 \to \mathbb{R}^3$.
3) Two knots K_1, K_2 are called *isotopically equivalent*, $K_1 \sim K_2$, if there is a family of homeomorphisms $\{\varphi_t\}_{t\in[0,1]}$, $\varphi_t\colon \mathbb{R}^3 \to \mathbb{R}^3$, such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1(K_1) = K_2$. (And similar for links.)

Example 8.2.

1) The unknot

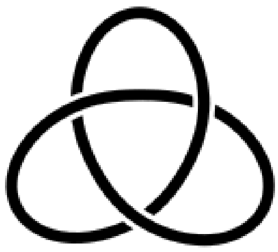


2) A trivial link, consisting of three unknots



3) The trefoil knot comes in two versions.

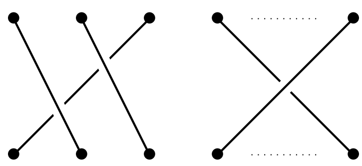
(a) left trefoil



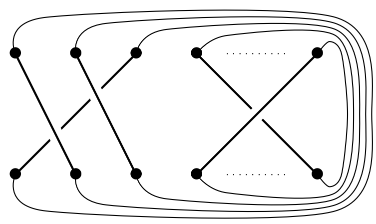
(b) right trefoil

Remark 8.3.

1) By Alexander's Theorem (1923), each link can be gotten by "closing" a braid.

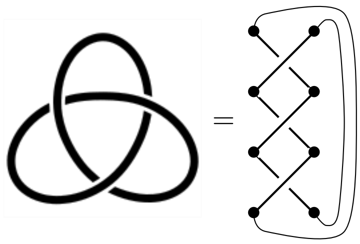


n-braid

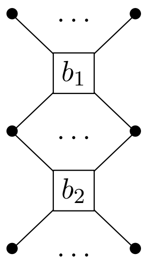


closed n-braid

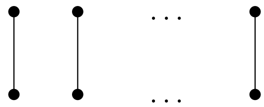
A famous example is the trefoil knot:



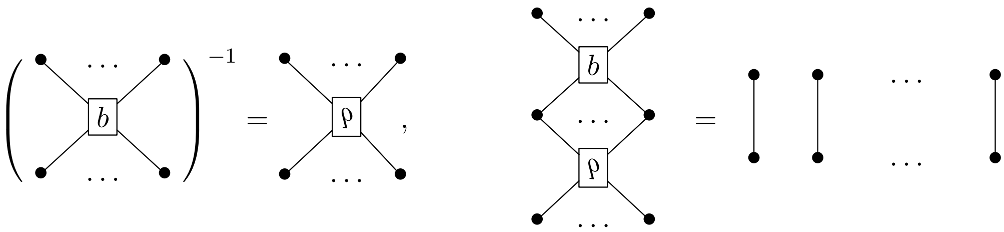
2) Braids can be multiplied by concatenation:



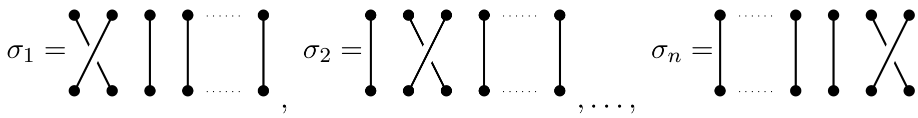
with neutral element



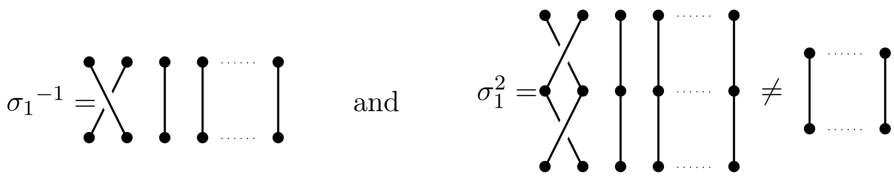
This gives actually a group structure where the inverse is given by taking the mirror image:



3) The braid group is generated by the elements $\sigma_1, \dots, \sigma_n$, where



Note that

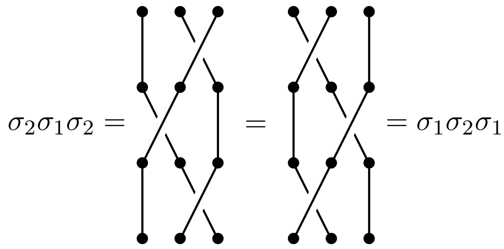


Definition 8.4.
The *n-string braid group* B_n is given by the generators $\sigma_1, \dots, \sigma_{n-1}$ and the following relations.

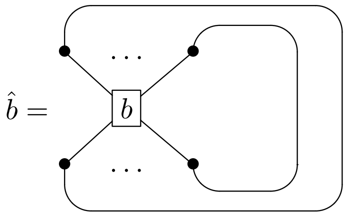
1) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ for all $i = 1, \ldots, n-2$
2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ whenever $|i-j| \geq 2$

Remark 8.5.

1) The relations from Definition 8.4 are clearly satisfied for the diagrammatic braids:



2) If we also require $\sigma_i^2 = 1$ for all i, then we don't distinguish under- and over-crossings and are just left with the permutation group S_n.
3) For a braid b its closure $\hat{b}$ is the link given by



Theorem 8.6 (Alexander 1923)**.**
Every link can be obtained as the closure $\hat{b}$ of some braid b.

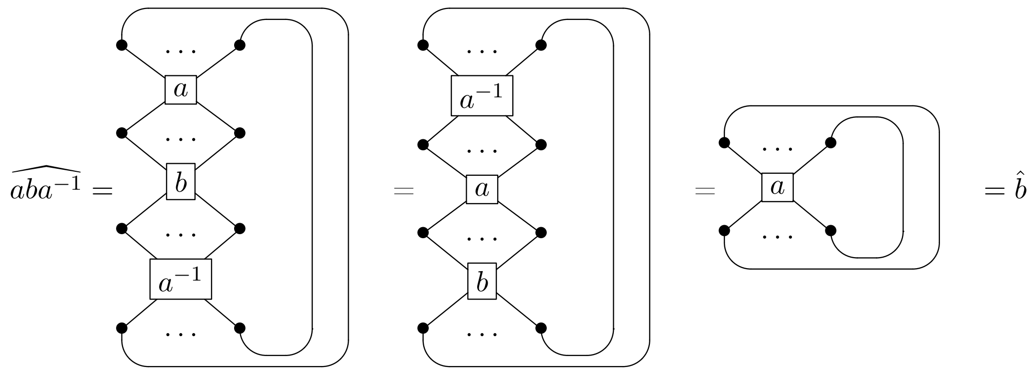
Remark 8.7.
There are modern effective algorithms for calculating the braid for a given link, due to Yamada (1987) and Vogel (1990).

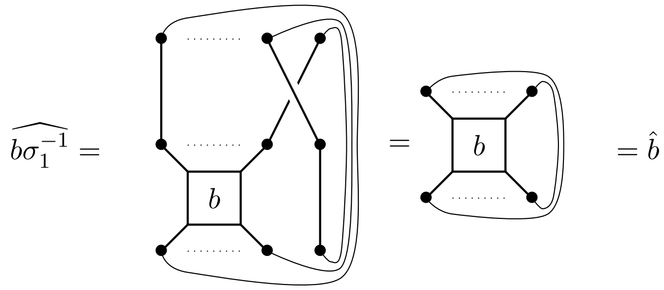
Theorem 8.8 (Markov 1935)**.**
Two braids give under closing the same link, if they can be transformed from one to another by a sequence of *type* I and *type* II *Markov moves*, where

1) type I: conjugation within B_n i.e.
$$b \leftrightsquigarrow a^{-1}ba \qquad (a, b \in B_n)$$
2) type II: for $b \in B_n$
$$B \ni b \leftrightsquigarrow b\sigma_n^{\pm 1} \in B_{n+1}.$$

Remark 8.9.

1) It is easy to see that those moves do not change the closure:





That those two types of moves suffice is not so clear; it can, for example, be deduced from Reidemeister's theorem on moves for diagrams of links.

2) Markov's theorem implies: if we have a function on braids which is invariant under Markov moves, then it is an invariant for knots/links.

Proposition 8.10.
Let $TL_n(\lambda)$ be the Temperley-Lieb algebra generated by $e_1, \dots, e_{n-1}$; we define

$$\psi_i := 1 - (1+t)e_i \in TL_n(\lambda) \qquad \text{for } i = 1, \dots, n-1.$$

Then we have for any choice of $t \in \mathbb{C}$ that

$$\psi_i \psi_j = \psi_j \psi_i \qquad \text{for } |i-j| \geq 2.$$

Moreover for t with $\lambda^{-1} = 2 + t + t^{-1}$ we have

$$\psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1} \qquad \text{for all } i = 1, \dots, n-2.$$

Furthermore ψ_i is invertible in $TL_n(\lambda)$ with $\psi_i^{-1} = 1 - \left(1 + \frac{1}{t}\right) e_i$. Hence, for t with

$\lambda^{-1} = 2 + t + t^{-1}$, we have a representation

$$\begin{aligned}\beta_t\colon B_n &\to TL_n(\lambda)\\ 1 &\mapsto 1\\ \sigma_i &\mapsto 1-(1+t)e_i\\ \sigma_i^{-1} &\mapsto 1-\left(1+\tfrac{1}{t}\right)e_i.\end{aligned}$$

Proof. Firstly, the fact that $\psi_i\psi_j = \psi_j\psi_i$ for $|i-j| \geq 2$ is clear, since it holds for the e_i. Next we calculate

$$\begin{aligned}\psi_i\psi_{i+1}\psi_i &= [1-(1+t)e_i][(1-(1+t)e_{i+1}][1-(1+t)e_i]\\ &= [1-(1+t)e_i][1-(1+t)e_{i+1}-(1+t)e_i+(1+t)^2e_{i+1}e_i]\\ &= 1-(1+t)e_{i+1}-(1+t)e_i+(1+t)^2e_{i+1}e_i-(1+t)e_i+(1+t)^2e_ie_{i+1}\\ &\quad+(1+t)^2\underbrace{e_i^2}_{e_i}-(1+t)^3\underbrace{e_ie_{i+1}e_i}_{\lambda e_i}\\ &= 1-(1+t)e_i[1+1-(1+t)+(1+t)^2\lambda]-(1+t)e_{i+1}+(1+t)^2e_ie_{i+1}\\ &\quad+(1+t)^2e_{i+1}e_i.\end{aligned}$$

In order that this is the same as $\psi_{i+1}\psi_i\psi_{i+1}$, we need that the expression above is invariant under the exchange of i and $i+1$, i.e. we need

$$1 = 1+1-(1+t)+(1+t)^2\lambda \qquad \text{or equivalently} \qquad t = (1+t)^2\lambda,$$

which means

$$\lambda^{-1} = \frac{(1+t)^2}{t} = 2+t+\frac{1}{t}.$$

Finally note that

$$[1-(1+t)e_i][1-(1+\tfrac{1}{t})e_i] = 1-(1+t+1+\tfrac{1}{t})e_i+(1+t)(1+\tfrac{1}{t})e_i^2 = 1$$

q.e.d.

Remark 8.11.
We can now compose β_t with the trace τ_n on $TL_n(\lambda)$ and check how this function on the braid group behaves with respect to Markov moves.

1) Invariance under move I is clear since

$$\tau_n\left(\beta_t\left(aba^{-1}\right)\right) = \tau_n\left(\beta_t(a)\beta_t(b)\beta_t\left(a^{-1}\right)\right) = \tau_n\Big(\underbrace{\beta_t\left(a^{-1}\right)\beta_t(a)}_{=\beta_t(a^{-1}a)=1}\beta_t(b)\Big) = \tau_n(\beta_t(b)).$$

2) It is not invariant under move II but changes by a simple factor: by property (3) of Proposition 7.6 for our trace τ_n (this is usually called the "Markov property") we have for $b \in B_n$

$$\tau_{n+1}\left(\beta_t\left(b\sigma_n^{\pm 1}\right)\right) = \tau_{n+1}\left(\underbrace{\beta_t(b)}_{\text{word in } 1, e_1, \ldots, e_{n-1}} \underbrace{\beta_t\left(\sigma_n^{\pm 1}\right)}_{\text{word in } 1, e_n}\right) = \tau_n(\beta_t(b)) \cdot \tau_{n+1}\left(\beta_t\left(\sigma_n^{\pm 1}\right)\right).$$

Let us calculate this factor (which is independent of i). We have

$$\beta_t(\sigma_1) = 1 - (1+t)e_1 \qquad \text{and hence} \qquad \tau_2(\beta_t(\sigma_1)) = 1 - (1+t)\lambda.$$

From the last proof we know

$$\lambda^{-1} = \frac{(t+1)^2}{t} \qquad \text{or equivalently} \qquad \lambda = \frac{t}{(t+1)^2}.$$

This yields

$$\tau_2(\beta_t(\sigma_1)) = 1 - (1+t)\lambda = 1 - (1+t)\frac{t}{(t+1)^2} = \frac{1}{1+t} = \frac{t^{-\frac{1}{2}}}{\sqrt{t} + \frac{1}{\sqrt{t}}},$$

and analogously (note that we get the inverse by replacing t with $1/t$)

$$\tau_2\left(\beta_t\left(\sigma_1^{-1}\right)\right) = \frac{1}{1+\frac{1}{t}} = \frac{t^{\frac{1}{2}}}{\sqrt{t} + \frac{1}{\sqrt{t}}}.$$

Definition 8.12.
The *Jones polynomial* for a link $\hat{b}$ with $b \in B_n$ is defined as

$$V_{\hat{b}}(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-1} \sqrt{t}^{\operatorname{wr}(b)} \tau_n(\beta_t(b)),$$

where $\operatorname{wr}(b)$ is the *writhe* of b, i.e. the number of $\sigma_i^{\pm 1}$ occurring in b, counted with sign. (In the diagram for the braid b, this counts the number of under- and overcrossing with sign.)

Remark 8.13.

1) For $\in B_n$ with

$$b = \sigma_{i(1)}^{k(1)} \cdots \sigma_{i(m)}^{k(m)}, \qquad i(1), \ldots i(m) \in \{1, \ldots, n-1\}, \quad k(1), \ldots, k(m) \in \{\pm 1\}$$

the writhe

$$\operatorname{wr}(b) = \sum_{i=1}^{m} k(i) \qquad (\text{for } m = 0\text{: } \operatorname{wr}(1) = 0)$$

is independent of the representation of b as a product in generators and inverses, because it does not change under the relations

- $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$,
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$,
- $\sigma_i\sigma_i^{-1}$.

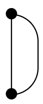
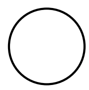
2) The writhe does not change under Markov moves of type I: $\mathrm{wr}\left(aba^{-1}\right) = \mathrm{wr}(b)$, since the contributions of a and a^{-1} cancel each other. For a type II move we have

$$\mathrm{wr}(b\sigma_n) = \mathrm{wr}(b) + 1 \qquad \text{and} \qquad \mathrm{wr}\left(b\sigma_n^{-1}\right) = \mathrm{wr}(b) - 1.$$

3) Hence the Jones polynomial does not change under moves of type I and II, and thus is an invariant for links.

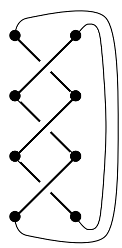
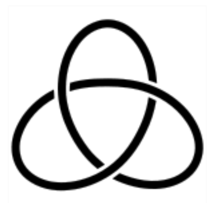
Example 8.14.

1) For the unknot

$\hat{b} =$  $=$ 

we have $b = 1$ and hence $V_{\hat{b}}(t) = 1$.

2) For the left-handed trefoil we have the diagram

$\hat{b} =$  $=$ 

where $b = \sigma_1^3 \in B_2$ and hence

$$V_{\text{left-trefoil}}(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) \sqrt{t}^3 \tau_2\left(\beta_t\left(\sigma_1^3\right)\right).$$

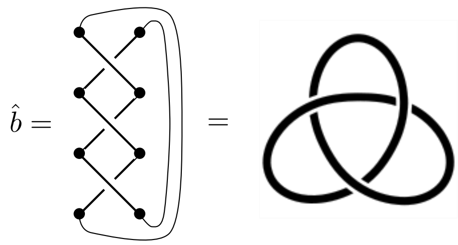
Now we have

$$\begin{aligned}
\tau_2\left(\beta_t\left(\sigma_1^3\right)\right) &= \tau_2\left(\beta_t\left(\sigma_1\right)^3\right) \\
&= \tau_2\left(\left(1-(1+t)e_1\right)^3\right) \\
&= \tau_2\left(\left(1-e_1\right)-te_1\right)^3\Big) \\
&= \tau_2\left(\left(1-e_1\right)-t^3e_1\right)\Big) \\
&= \tau_2\left(\left(1-(1+t^3)e_1\right)\right) \\
&= 1-(1+t^3)\lambda \\
&= 1-(1+t^3)\frac{t}{(1+t)^2} \\
&= 1-t(1-t+t^2)\frac{t^{-\frac{1}{2}}}{\sqrt{t}+\frac{1}{\sqrt{t}}},
\end{aligned}$$

resulting finally in

$$V_{\text{left-trefoil}}(t) = t + t^3 - t^4.$$

3) Analogously we can calculate the Jones polynomial for the right trefoil



where $b = \sigma_1^{-3}$. Note that in general the Jones polynomial for b and b^{-1} are related by replacing t by $\frac{1}{t}$, Hence we have

$$V_{\text{right-trefoil}}(t) = \frac{1}{t} + \frac{1}{t^3} - \frac{1}{t^4} \neq V_{\text{left-trefoil}}(t)$$

This shows that the left-handed and right-handed trefoil are different knots! (Before the Jones polynomial, there was no easy invariant to distinguish those two knots.)

9 The Standard Invariant of Subfactors and an Informal Introduction to Planar Algebras

For a subfactor $N \subset M$ we have now one invariant, the index $[M : N]$. This raises the questions:

- How many different subfactors for a fixed index exist?
- Are there finer invariants?

The first question, asked in this generality, has the answer "way too many", as one can produce (for example, by taking tensor products) an abundance of other II_1 factors out of given ones without changing the index. Hence, we want to ask the first question in the hyperfinite case, i.e. for $\mathcal{R} \cong N \subset M \cong \mathcal{R}$.

Note that we understood $[M : N]$ via the Jones projections coming from the basic construction; those Jones projections were actually elements of some relative commutants, namely $e_n \in M_{n-1} \cap M'_{n-3}$. We consider now more generally the data given by the relative commutants of our tower construction $N \subset M \subset M_1 \subset M_2 \subset \dots$; it turns out that the relevant relative commutants are given by

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \dots \\ & & \cup & & \cup & & \cup & \\ & & \mathbb{C} = M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \dots \end{array} \tag{9.1}$$

Note that for the case of finite index, $[M : N] < \infty$, we have (see Corollary 4.6):

- all relative commutants are finite dimensional, hence those inclusions are described by Bratteli-diagrams.

In "nice" cases, we expect/hope to recover the original subfactors as an inductive limit of relative commutants.

$$\begin{array}{ccc} M_\infty & = & \overline{\bigcup_k M' \cap M_k} \\ \cup & & \cup \\ N_\infty & = & \overline{\bigcup_k N' \cap M_k} \end{array}$$

(Then, M_∞ and N_∞ are necessarily hyperfinite.) More on this in the next section.

The collection of relative commutants in Equation (9.1) is called the *standard invariant* $\mathcal{G}_{N,M}$ of the subfactor $N \subset M$. There are various ways of axiomatizing those data in an abstract way:

→ paragroup (Ocneanu)

→ λ-lattice (Popa)

→ planar algebra (Jones)

We will in the following concentrate on the "planar algebra" approach of Jones.

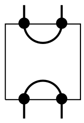
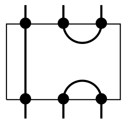
Let us for the moment only consider one of the two rows in Equation (9.1). We have there an inclusion of finite-dimensional vector-spaces $(P_n := N \cap M_{n-1})_{n \geq 0}$,

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \dots \\ \Vert & & \Vert & & \Vert & & \Vert & \\ P_0 & \subset & P_1 & \subset & P_2 & \subset & P_3 & \dots \end{array},$$

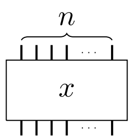
which contain also the Jones projections:

$$\begin{aligned} &P_0 = \mathbb{C} \\ &P_1 \supset \{1\} \\ &P_2 \supset \{1, e_1\} \\ &P_3 \supset \{1, e_1, e_2\} \\ &\quad\vdots \end{aligned}$$

For the e_i's, we have, via $e_i = \frac{1}{\alpha} E_i$, a diagrammatic representation:

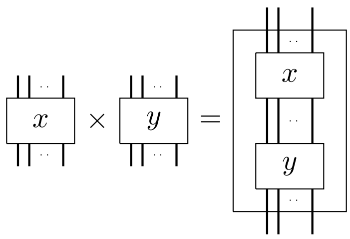
$E_1 =$  $\in P_2$, $E_2 =$  $\in P_3$, $\cdots$

Motivated by this, we want to represent all elements in the relative commutants in terms of diagrams. We will think of a general element $x \in P_n$ an n-box (n strings on top and n strings on bottom)

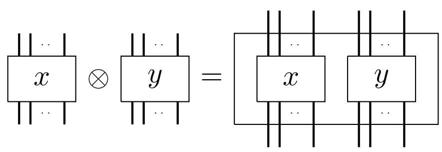


and on those element we have operations given by diagrams:

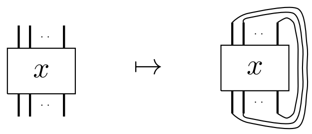
- multiplication $\times\colon P_n \times P_n \to P_n$ by



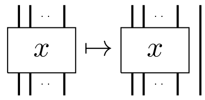
- tensor product $\otimes\colon P_n \times P_n \to P_n$ by



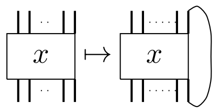
- trace tr: $P_n \to P_0 \cong \mathbb{C}$ given for an $x \in P_n$ and an corresponding n-box by



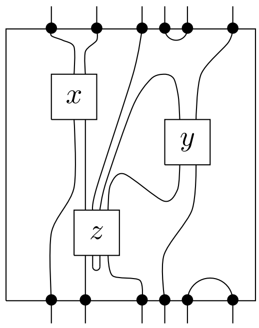
- embedding $P_n \to P_{n+1}$



- conditional expectation $P_n \to P_{n-1}$

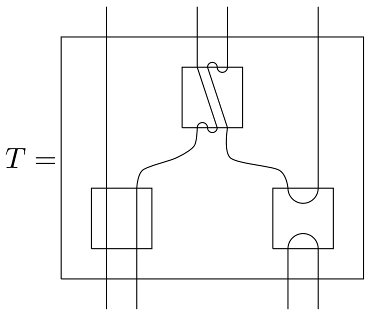


We can combine all those operations to get actions of arbitrary *planar tangles* on our *planar algebra* $\bigoplus_{n\geq 0} P_n$. For instance consider the planar tangle

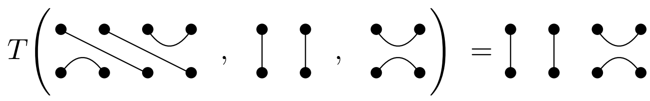


This should be thought of as a multi-linear mapping $P_3 \times P_2 \times P_4 \to P_4$, where we plug in elements $x \in P_3$, $y \in P_2$, $z \in P_4$ and then get an element in P_4.

Here is a concrete example of this for another tangle T, where we have used some the elements from the Temperley-Lieb algebras as input.



In this case we have



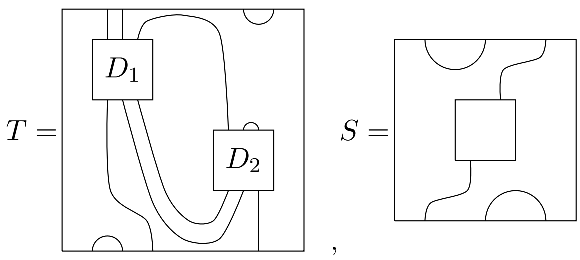
Since we are dealing with two sequences of inclusions in the standard invariant, we need two versions of the P_n, $P_{n,+}$ and $P_{n,-}$, and the actions of the planar tangles has to take this into account, by a checker board shading. We will ignore this most of the time.

Summing up we have

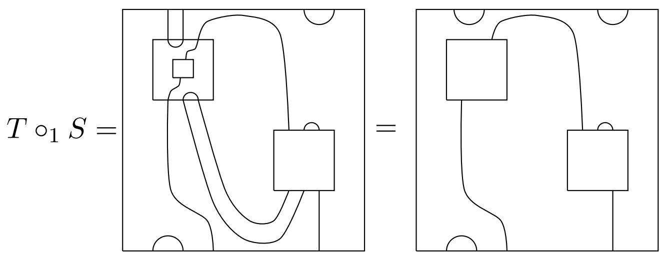
planar tangles $\equiv$ multilinear mappings
planar algebra $\equiv$ collection of vector spaces together with multilinear mappings for all planar tangles

Abstractly the planar tangles correspond to an "operad structure" and the planar algebras are "algebras over the operad of planar tangles". The main point of all this is that we have a composition of planar tangles, which must be represented by multilinear mappings.

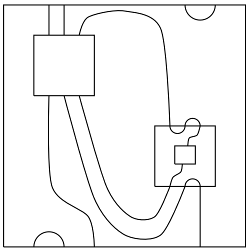
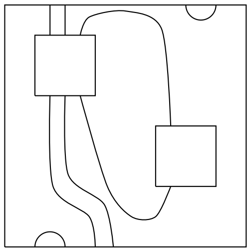
Example 9.1.
Here is an example for composition of tangles. Consider two tangles T and S.



We want to compose T with S. We need to specify in which of the slots of T we insert S. Of course, the number of points has to agree; in our case, both input boxes of T can take on S; hence we numbered those boxes as D_1 and D_2 and have two possible compositions.



and

$T \circ_2 S =$  $=$ 

The composition of planar tangles is kind of generalization of the composition of functions.

Definition 9.2.
A (unshaded) *planar algebra* $P = \bigoplus_{n \geq 0} P_n$ is a family of vector spaces P_n, such that for each planar tangle T there is a multi-linear map

$$Z_T \colon \bigtimes_{D \in \mathcal{D}_T} P_D \to P_{D_0},$$

($\mathcal{D}_T$ denotes here the inner boxes of T, D_0 is the outer box of T, and for a box D we have $P_D = P_n$ where $n = \frac{1}{2}\#$ marked points on D), subject to the requirements:

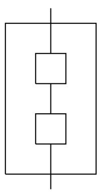
1) isotropy invariance (topological deformations of the tangles do not change the mappings)
2) naturality

$$Z_{T \circ S} = Z_T \circ Z_S \qquad \text{for all tangles } T, S \text{ with valid composition } T \circ S;$$

note that the left hand side is a composition of tangles, on the right hand side we have a composition of multi-linear maps.

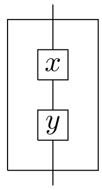
Remark 9.3.

1) Usually we will simplify our drawings by bundling parallel strings to one string; the number n of strings should then be clear from the context.
2) The P_n have a canonical algebra structure given by the tangle



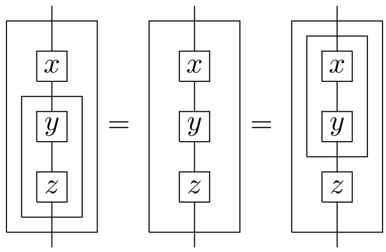
(here every line consists of n strings). Then for $x, y \in P_n$ we put

$$x \cdot y := Z_{T_n}(x, y) =$$



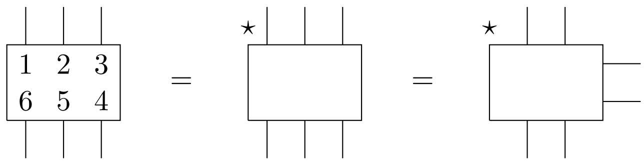
Associativity follows by naturality

$x \cdot (y \cdot z) =$



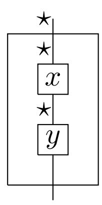
$= (x \cdot y) \cdot z$

3) We actually have to remember the numbering of the strings around our boxes. Up to now we have oriented our boxes in a standard way, where the numbering starts at the upper left corner of the box; however, it will be advantegeuous if we allow our strings also to move around the box; then we put a $*$ at the position of the first string. Here is an example for this.

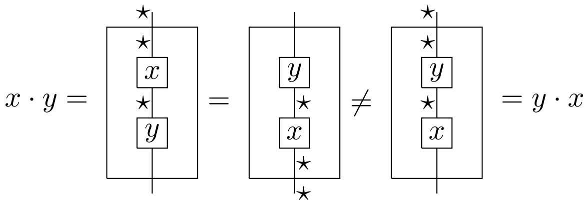


Hence our multiplication is given more precisely as follows:

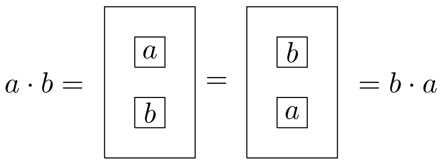
$x \cdot y =$



In this representation it becomes also clear that in general the multiplication in non-commutative, namely we have



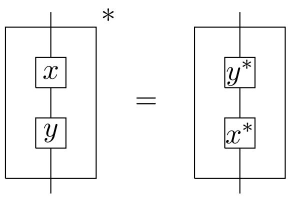
However, for $n = 0$ we have no strings to put a star on, hence we have



Thus P_0 is a commutative algebra. In the subfactor setting we have $P_0 \cong \mathbb{C}$.

4) For a *subfactor planar algebra* (SPA), we require much more (reflecting properties which we find for the relative commutants in the Jones tower)

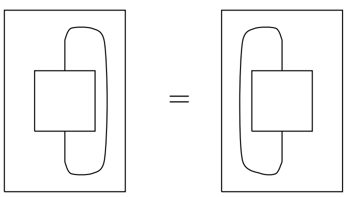
   a) involution: we have an involution $*$ on each P_n, which is compatible with the reflecting of tangles



   i.e. $(x \cdot y)^* = y^* \cdot x^*$.

   b) dimension restrictions: $\dim(P_0) = 1$ and $\dim(P_n) < \infty$ for all n.

   c) sphericality: we require that the left and the right trace are the same:



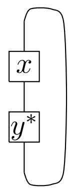
$P_1 \to P_0 \cong \mathbb{C}$ $\qquad$ $P_1 \to P_0 \cong \mathbb{C}$

   This means that we have isotropy actually on the sphere, not just on the plane.

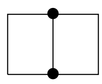
d) positivity of the inner product defined by

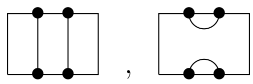
$$\langle \cdot, \cdot \rangle \colon P_n \times P_n \to P_0, \quad (x, y) \mapsto \mathrm{tr}(y^* x).$$

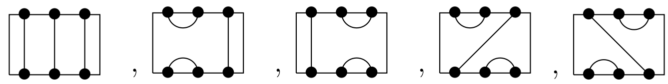
Pictorially the inner product is given by

$\langle x, y \rangle =$ 

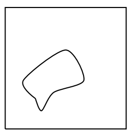
5) Note that tangles with no input discs will be thought of as acting on $\emptyset$ and will thus give an element in P. Hence, such tangles will be contained in any planar algebra. So all the Temperley-Lieb diagrams (which are tangles without input disc) are contained in any planar algebra.

$TL_1 =$ span $\Bigg\{$  $\Bigg\} \in P_1$

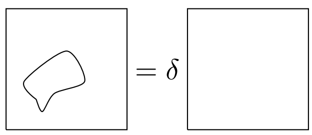
$TL_2 =$ span $\Bigg\{$  $\Bigg\} \in P_2$

$TL_3 =$ span $\Bigg\{$  $\Bigg\} \in P_3$

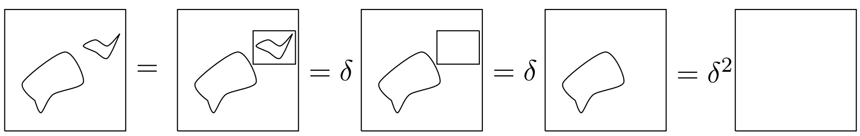
Note that in the case where $\dim P_0 = 1$ we also have:

 $\in P_0 \cong \mathbb{C}$

Hence the closed circle is given by some $\delta \in \mathbb{C}$, i.e.



but then by naturality



i.e. each closed circle gives a factor δ.

6) In the non-subfactor case, we can also consider tangles with odd numbers of incoming strings or we can also consider directed strings.

Example 9.4.

1) The *zero planar algebra* is given by

$$P_n = \text{ arbitrary vector spaces } \forall n \qquad \text{and} \qquad Z_T = 0 \quad \forall \text{ tangles } T.$$

2) The *trivial planar algebra* is given by

$$P_n = \mathbb{C} \quad \forall n, \qquad Z_T = \text{product map for any tangle}$$

3) More interesting is the *tensor planar algebra*. Let V be a finite-dimensional vector space and set $P_n = V^{\otimes n}$. Let $e_1, \dots, e_k$ be a basis of V, then

$$\{e_{i(1)} \otimes \cdots \otimes e_{i(n)} \colon i(1), \dots, i(n) \in \{1, \dots, k\}\}$$

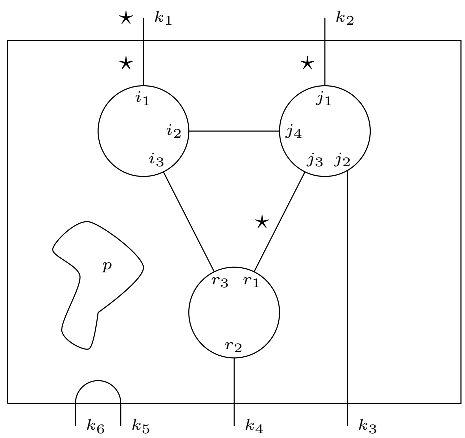
is a basis of $V^{\otimes n}$. Given a tangle T, we have to define its action on elements

$$v \in \bigtimes_{D \in \mathcal{D}_T} P_D.$$

By multi-linearity, it suffices to define this for basis elements as inputs; for those we set

$$T(e_{i_1} \otimes \cdots \otimes e_{i_r}, e_{j_1} \otimes \cdots \otimes e_{j_s}, \dots) = \sum_k \delta^k_{i,j,\dots}(T) e_{k_1} \otimes \cdots \otimes e_{k_t}$$

where $\delta^k_{i,j,\dots}(T)$ is 1 if all connected indices agree, otherwise 0. Indices are assigned in a cyclic way to their discs; closed loops without input contribute with a factor k, corresponding to summing over a basis. For example consider

$T =$ 

then

$$T(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}, e_{j_1} \otimes e_{j_2} \otimes e_{j_3} \otimes e_{j_4}, e_{r_1} \otimes e_{r_2} \otimes e_{r_3}) = \sum_{\substack{k_1,\dots,k_6 \\ p}} \delta^k_{i,j,r,p}(T) e_{k_1} \otimes \cdots \otimes e_{k_6},$$

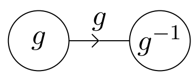
where

$$\delta^k_{i,j,r,p}(T) = 1 \iff k_1 = i_1, \quad k_2 = j_1, \quad k_3 = j_2 \quad k_4 = r_2, \quad k_5 = k_6.$$

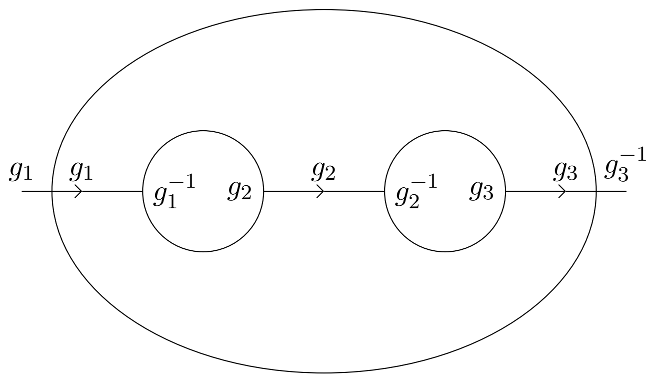
and the loop gives no condition on p, hence $\sum_p$ gives a factor k. Note that the "planarity" of the constraints is not necessary here. Without planarity this corresponds to *Penrose's graphical notation for tensors*; if we allow directed strings, then we can also distinguish between covariant and contravariant indices. In a sense we are looking here at spaces of tensors which are closed under "planar" contractions.

4) Let us also look at an example where the planarity of the tangles is clearly relevant: consequences from group identities. Take tangles with orientation, assign group elements to strings and identify

disc $\equiv$ relation by reading attached group elements in cyclic order, taking elements or its inverses according to orientation.



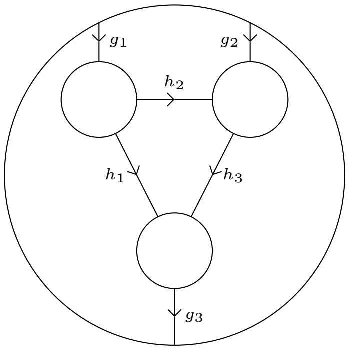
Then the relation at the output disc is a consequence of the relations at input discs. Here is an easy example.



This encodes the implication

$$\left.\begin{aligned} g_2 g_1^{-1} &= 1 \\ g_3 g_2^{-1} &= 1 \end{aligned}\right\} \implies g_1 g_3^{-1} = 1.$$

Another more interesting example is given by the tangle



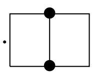
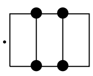
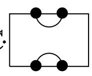
which corresponds to the following implication.

$$\left.\begin{array}{l} g_1^{-1}h_2h_1 = 1 \\ g_2^{-1}h_3h_2^{-1} = 1 \\ g_3h_1^{-1}h_3^{-1} = 1 \end{array}\right\} \implies g_1g_2g_3^{-1} = 1$$

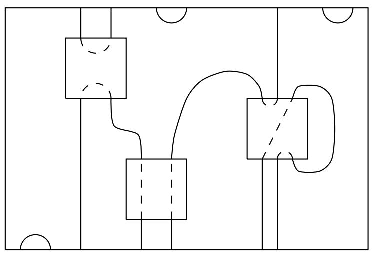
5) Temperley-Lieb planar algebra:

$$P_n = TL_n = \text{span}\{\ NC\text{-pairings on } 2n \text{ points }\},$$

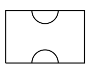
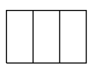
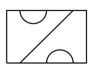
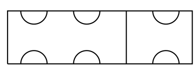
We have

$TL_1 = \mathbb{C}\cdot$  $\qquad TL_2 = \mathbb{C}\cdot$  $+\mathbb{C}\cdot$ 

The action of a tangle T on these diagrams is given by plugging in the TL-diagrams in the input discs and then erase the boxes. Consider the following example

$T =$ 

thus we have

$T = \Big($  ,  ,  $\Big) =$ 

We also have to assign a value $\delta \in \mathbb{C}$ to loops. The positive definiteness of the trace gives restrictions on δ (namely those for the index, with $\delta^2 = [M : N]$).

10 How to Make a Factor out of a Planar Algebra

Theorem 10.1.
Let $P = (P_{n,\pm})_{n\geq 0}$ be a shaded subfactor planar algebra. Then there exists a subfactor $N \subset M$ such that P is the planar algebra of this subfactor, i.e $P_{n,\pm}$ are the relative commutants of the tower from $N \subset M$.

This theorem was proved by

- Popa 1995 (by using λ-system instead of planar algebras)
- Guionnet, Jones, Shlyakhtenko 2007 (by changing the trace on the planar algebra)
- Jones, Shlyakhtenko, Walker 2008 and Kodiyalam, Sunder 2008 (by changing the multiplication on the planar algebra)

Remark 10.2.
One of the main issues is how to construct a *factor* out of a subfactor planar algebra (SPA). One should note that in the axioms of a SPA there must be hidden some non-obvious restrictions on P, if a SPS is the same as the relative commutants of a subfactor. Here is an example of this. Consider a subfactor $N \subset M$. Then we have seen that

$$[M : N] \geq \sum_i \frac{1}{\tau_M(p_i)}$$

for any family (p_i) of orthogonal projections in $N' \cap M$ with $\sum_i p_i = 1$. Let $(p_i)_{i=1}^k$ be a maximal family of orthogonal projections in $N' \cap M$ with $\sum_{i=1}^k p_i = 1$, then on one hand we have $\dim(N' \cap M) \leq k^2$, whereas on the other hand, by Cauchy-Schwartz, we also have the implication

$$\sum_{i=1}^k \tau(p_i) = 1 \implies \sum_{i=1}^k \frac{1}{\tau(p_i)} \geq k^2.$$

Putting this together yields

$$\dim(N' \cap M) \leq k^2 \leq \sum_{i=1}^k \frac{1}{\tau(p_i)} \leq [M : N].$$

If we consider now the tower $N \subset M \subset M_1 \subset M_2 \dots$ then we get

$$\dim(N' \cap M_k) \leq [M_k : N] = [M : N]^{k+1}.$$

This shows that $\dim(P_n)$ for our SPA cannot be arbitrary, but must have moderate growth. Note that there is no restriction like this in the definition of SPA; this must be a consequence of planarity and positivity.

Example 10.3.

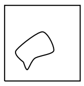
1) If we take the Temperley-Lieb planar algebra, i.e.

$$P_n = \text{span}\{NC\text{-pairings of } 2n \text{ points}\},$$

then

$$\dim(P_n) \leq \#NC_2(2n) = \frac{1}{n+1}\binom{2n}{n} \sim 4^n.$$

Here the growth is okay and we can have all NC-pairings linearly independent. Note that this depends on the trace, which is determined via


$= \delta$

2) If we want to take $P_n = \text{span}\{\text{all pairings of } 2n \text{ points}\}$ (and some rule for the calculation of crossing strings) then we have

$$\dim(P_n) \leq \#\mathcal{P}_2(2n) = (2n-1)!! = (2n-1)\cdot(2n-3)\ldots 5\cdot 3\cdot 1.$$

Those are the moments of a classical Gauss distribution and not exponentially bounded. Hence the growth is too fast and we cannot have all pairings linearly independent; i.e., our inner product, if positive, must necessarily have some non-trivial kernel.

Remark 10.4.
Consider now SPA $P = (P_n)_{n\geq 0}$ (unshaded). How can we get a II_1 factor M out of this? A first canonical idea is as follows: we have, by the planar algebra operations, embeddings $P_0 \subset P_1 \subset P_2 \subset P_3 \subset \ldots$ of finite dimensional C*-algebras, equipped with compatible traces; hence we can take the inductive limit and then do the GNS construction with respect to the trace τ, resulting in

$$M = \overline{\bigcup_{n\geq 0} P_n}^{\tau}.$$

This is now a finite von Neumann algebra, but there is no reason that this should be a factor in general. So one needs a new idea.

Before we look on this, let us remark that there are nice cases where the above direct GNS construction yields a factor. In this case the factor M is clearly hyperfinite. The Temperley-Lieb case is a special example for this; this was actually the construction of subfactors for the allowed indices less than 4 by Jones. Before we go to the case of general SPA we want to treat the case of the Temperley-Lieb planar algebra. There we have so much concrete information about generators of TL (namely, the Jones projections), that one can show factoriality of M – by proving that there is only one normal trace on M.

Lemma 10.5.
A cyclically reduced word in the e_i from TL is of the form

$$w = e_{i_1} e_{i_2} \cdots e_{i_n} \qquad \text{with } |i_j - i_k| \geq 2 \text{ whenever } j \neq k. \tag{10.1}$$

Hence any trace on TL is determined by its action on words of the form (10.1).

Proof. We only show by an example how one can bring arbitrary words in TL into the form (10.1). Consider the word $w = e_1 e_2 e_3$. This is linearly reduced but not cyclically:

$$e_1 e_2 e_3 \quad \rightsquigarrow \quad e_1 e_2 e_3 e_3 \quad \rightsquigarrow \quad e_3 e_1 e_2 e_3 \quad \rightsquigarrow \quad e_1 \underbrace{e_3 e_2 e_3}_{\lambda e_3} \quad \rightsquigarrow \quad \lambda e_1 e_3,$$

i.e. for any trace tr we have $\operatorname{tr}(e_1 e_2 e_2) = \lambda \operatorname{tr}(e_1 e_3)$. q.e.d.

Lemma 10.6.
Let $w_1 = e_{i_1} e_{i_2} \ldots e_{i_m}$ and $w_2 = e_{j_1} e_{j_2} \ldots e_{j_m}$ be two words of the form (10.1) with the same length m. Then there exists a unitary $u \in A_\infty = \operatorname{alg}\{e_n \colon n \in \mathbb{N}\}$ such that $u w_1 u^* = w_2$. Hence for any trace we have

$$\operatorname{tr}(w_2) = \operatorname{tr}(u w_1 u^*) = \operatorname{tr}(w_1).$$

Proof. Again, we just do a typical example. Assume we want to see that for any trace $\operatorname{tr}(e_1 e_3) = \operatorname{tr}(e_1 e_4)$. For this, we want a unitary $u \in \operatorname{alg}\{e_3, e_4\}$ with $u e_3 u^* = e_4$. Such a u commutes with e_1, and thus

$$\operatorname{tr}(e_1 e_4) = \operatorname{tr}(e_1 u e_3 u^*) = \operatorname{tr}(u e_1 e_4 u^*) = \operatorname{tr}(e_1 e_3).$$

In order to find u, it suffices to have a partial isometry $v \in \operatorname{alg}\{e_3, e_4\}$ with $vv^* = e_3$ and $vv^* = e_4$. (Such a partial isometry can be dilated to a unitary.) But for this we take $v = \frac{1}{\sqrt{\lambda}} e_3 e_4$; then

$$vv^* = \frac{1}{\lambda} e_3 e_4 e_4 e_3 = \frac{1}{\lambda} e_3 e_4 e_3 = \frac{1}{\lambda} \lambda e_3 = e_3$$

and in the same way

$$v^* v = \frac{1}{\lambda} e_4 e_3 e_3 e_4 = e_4.$$

q.e.d.

Proposition 10.7.
The GNS construction $M = \overline{\bigcup_{n \geq 0} TL_n(\lambda)}^{\tau}$ yields a factor.
(Of course, we have to restrict here to values of λ for which $TL_n(\lambda)$ is a SPA, i.e., where the trace is positive - which is the case for $\lambda^{-1} \geq 4$ or $\lambda^{-1} = 4\cos^2(\pi/n)$ for $n \geq 3$.)

Idea of the proof. It suffices to show that for any trace tr on M we have $\mathrm{tr} = \tau$. By Lemma 10.5, it suffices to check equality on words w of the form (10.1). Let us again just do a telling example; say, we want to show that $\mathrm{tr}(e_1e_3) = \tau(e_1e_3)$. First, observe that, by Lemma 10.6, we know $\mathrm{tr}(e_1e_3) = \mathrm{tr}(e_1e_n)$ for any $n \geq 3$. But then, using the mean ergodic theorem (note that we invoke here only e_i which commute, hence this is a classical Bernoulli shift), we get

$$\mathrm{tr}(e_1e_3) = \mathrm{tr}\Big(e_1 \underbrace{\frac{e_3 + e_5 + e_7 + \cdots + e_{2n+1}}{n}}_{\xrightarrow[n\to\infty]{} \tau(e_3)1}\Big) = \mathrm{tr}(e_1\tau(e_3)1) = \mathrm{tr}(e_1)\tau(e_3).$$

Since we know that for our trace τ we have, by the Markov property, that $\tau(e_1e_3) = \tau(e_1)\tau(e_3)$, we have reduced the problem to showing that also $\mathrm{tr}(e_1) = \tau(e_1)$. But this follows in the same way as above: we have, by Lemma 10.6, that $\mathrm{tr}(e_1) = \mathrm{tr}(e_n)$ for all n, hence also

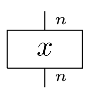
$$\mathrm{tr}(e_1) = \mathrm{tr}\Big(\underbrace{\frac{e_1 + e_3 + e_5 + \cdots + e_{2n-1}}{n}}_{\xrightarrow[n\to\infty]{} \tau(e_1)1}\Big) = \tau(e_1)$$

q.e.d.

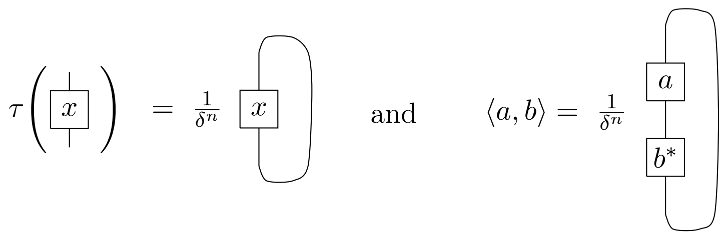
Remark 10.8.

1) In general this construction of taking the inductive limit does not work, since we cannot guarantee factoriality. In particular, we do not know which planar algebras can be realized as relative commutants of $N \subset M$ with $N, M \cong \mathcal{R}$ hyperfinite.

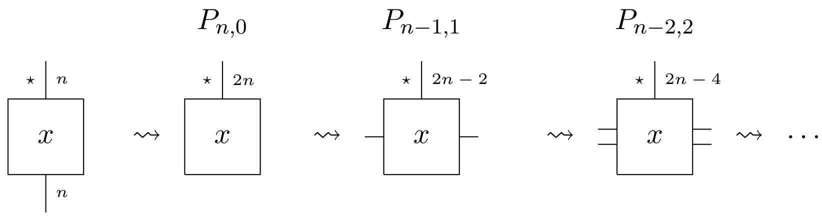
2) We consider now subfactor planar algebra $(P_n)_{n\geq 0}$ (we suppress shading and the distinction between $P_{n,+}$ and $P_{n,-}$), i.e.

$P_n = \mathrm{span}\Big\{$  $\Big\}$

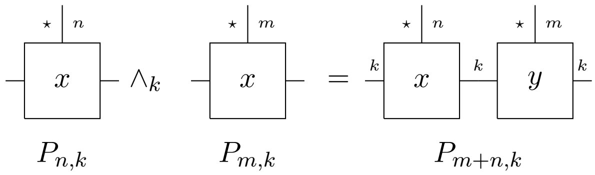
with trace τ and inner product on each P_n given by



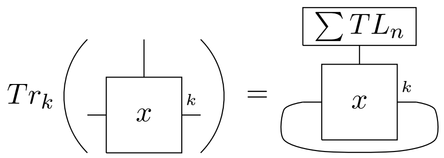
By assumption $P_0 \cong \mathbb{C}$, all P_n are finite dimensional and $\langle \cdot, \cdot \rangle$ is positive on all P_n. Guionnet, Jones and Shlyakhtenko considered different traces on P giving other possibilities for the GNS-construction. For $x \in P_n$ we can consider different pictures:



Depending on the choice of k in $P_{n,k}$ we change the multiplication to $\wedge_k$, given by



and the new trace is given by



where we sum over all possibilities with closing the upper strings with non-crossing pairings. If we equip $\bigoplus_{n=k}^{\infty} P_n$ with the multiplication

$$\wedge_k \colon P_m \times P_n \to P_{m+n-k}$$

and the trace Tr_k, then GJS showed that this gives a II_1 factor M_k and $M_0 \subset M_1 \subset M_2 \ldots$ is the Jones tower for $M_0 \subset M_1$, its relative commutants are the P_n, i.e., $P_n = M_0' \cap M_n$.

3) We will in the following instead look on equivalent constructions with other multiplication and trace, as given by Jones, Shlyakhtenko, Walker.

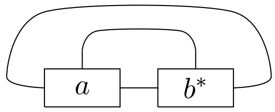
Definition 10.9.
For a given subfactor planar algebra $P = (P_n)_{n\geq 0}$ we put for each $k \geq 0$

$$\mathrm{Gr}_k(P) =: \bigoplus_{n\geq 0} P_{n+k} = \bigoplus_{n\geq 0} P_{n,k} \qquad \text{graded vector space}$$

with inner product given as follows: for $a \in P_{m+k}, b \in P_{n+k}$ we put

$$\langle a, b\rangle = 0 \quad \text{if } n \neq m$$

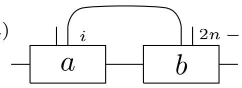
and

$$\langle a, b \rangle = \frac{1}{\delta^k}$$  if $m = n$.

We define a trace by

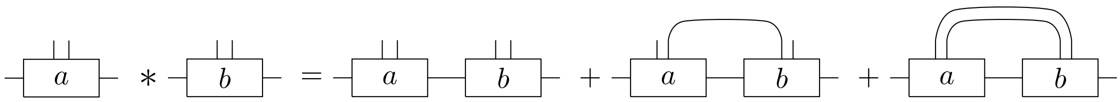
$$\operatorname{tr}(a) = \begin{cases} 0 & \text{if } a \in P_{n,k} \text{ and } n > 0 \\ \tau(a) & \text{if } a \in P_{0,k} \end{cases}.$$

Definition 10.10.
On $\mathrm{Gr}_k(P)$ we define a product for $a \in P_{m,k}, b \in P_{n,k}$ by

$$a * b = \sum_{i=0}^{\min(2m,2n)}$$ 

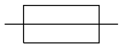
On $P_{n,k}$ we also have a $*$-structure (involution) coming from $P_{n,k} = P_{n+k}$.

We give a simple example for the multiplication, for $k = 1$ and $m = n = 1$.

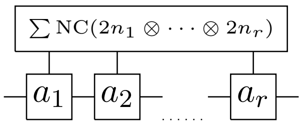


Remark 10.11.

1) $(\mathrm{Gr}_k(P), *, {}^*)$ is an associative $*$-algebra. Note that $a * (b * c) = (a * b) * c$ is given by all partial non-commutative contractions of the upper strings, where we do not pair strings from the same box.
2) Note that we now have $\langle a, b \rangle = \operatorname{tr}(a * b^*)$.
3) The unit of $\mathrm{Gr}_k(P)$ is given by

 $\in P_k$

4) Note that on $\mathrm{Gr}_k(P)$, for $a_i \in P_{n_i,k}$, the trace $\operatorname{tr}(a_1 * a_2 * \cdots * a_r)$ is given by the picture



where $NC(2n_1 \otimes \cdots \otimes 2n_r)$ are NC pairings of $2n_1 + \cdots + 2n_r$ which respect the boxes, i.e., which do not pair strings from the same box.

5) The idea is to take $M_k = \overline{\mathrm{Gr}_k(P)}^{\mathrm{tr}}$ acting by multiplication (via $*$) on $L^2(\mathrm{Gr}_k(P), \operatorname{tr})$; for this one has to see that the multiplication by a, $\xi \mapsto a * \xi$, is a bounded operator on L^2 for every a. Then one has to show:

- Each M_k is a II_1 factor.
- We have the inclusion

$$M_0 \subset M_1 \subset M_2 \subset M_3 \dots$$

- This inclusion of II_1 can be identified with the basic construction for $M_0 \subset M_1$.
- For all k we have

$$M_0' \cap M_k = P_{0,k} = P_k.$$

In the following we want to give the main ideas for those statements.

6) Note that with the changed multiplication we do not have any more inclusions of finite-dimensional algebras; hence the M_k will in general not be hyperfinite. Actually, one can show that at least in the case of finite depth the M_k are free group factors.

Notation 10.12.
In the following we put

$$\mathcal{H}_k = L^2(\mathrm{Gr}_k(P), \mathrm{tr}) = \overline{\mathrm{Gr}_k(P)}^{\langle \cdot, \cdot \rangle},$$

where

$$\langle a, b \rangle = \mathrm{tr}(a * b) \qquad \text{thus} \qquad \|\xi\|_k^2 = \mathrm{tr}(\xi * \xi^*).$$

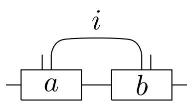
Proposition 10.13.
Let $a \in \mathrm{Gr}_k(P)$. Then the map

$$L_a\colon\ \mathrm{Gr}_k(P) \to \mathrm{Gr}_k(P), \quad b \mapsto a * b$$

is bounded with respect to $\| \cdot \|_k$, hence it extends to a bounded map on $\mathcal{H}_k$.

Proof. It suffices to consider $a \in P_{n,k} = P_{n+k}$ for some n. We write

$$L_a = \textstyle\sum_{i=0}^{2n} L^{(i)}, \text{ where } \quad L_a^{(i)}(b) = a *_i b =$$



so it suffices to control each $L_a^{(i)}$ $(i = 0, \dots, 2n)$.

Consider $b = \oplus_m b_m$ with $b_m \in P_{m,k}$. Note that

$$a *_i b = \sum_m \underbrace{a *_i b_m}_{\in P_{n+m-i,k}}$$

and hence

$$\|a *_i b\|_k^2 = \sum_m \|a *_i b_m\|_k^2. \tag{10.2}$$

We have to show that there is a constant $C > 0$ such that

$$\|a *_i b_m\|_k \leq C\|b_m\|_k \qquad \text{holds for all } m.$$

Because then we have

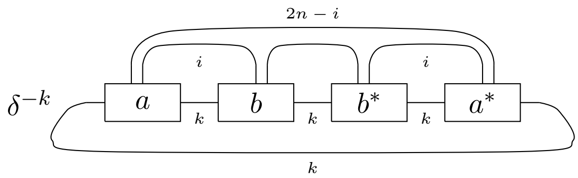
$$\|a *_i b\|_k^2 \leq \sum_m \|a *_i b_m\|_k^2 \leq \sum_m C^2\|b_m\|_k^2 = C^2\|b\|_k^2$$

and hence $\|L_a^{(i)}\| \leq C$.

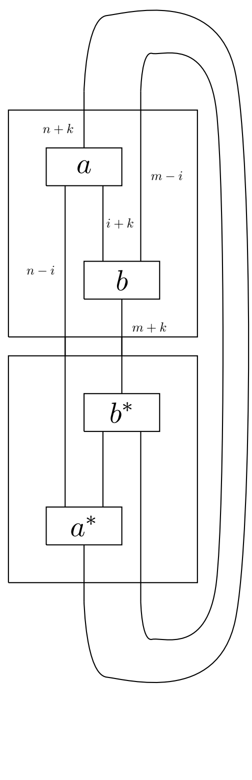
So we consider

$$\|a *_i b_m\|_k^2 = \langle a *_i b_m, a *_i b_m \rangle.$$

For notational simplicity, let us put $b = b_m$, then the inner product is given by the following picture.

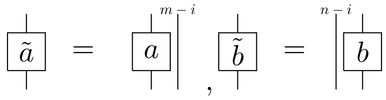


We redraw this picture as

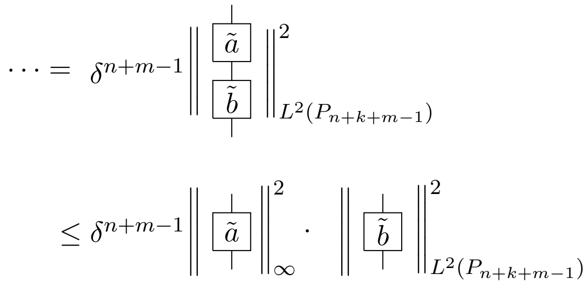


(10.3)

If we put



then we can continue the above calculation as follows

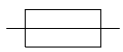


$$= \|a\|^2 \|b_m\|_k^2$$

q.e.d.

Definition 10.14.
Let (M_k, tr) be the finite von Neumann algebras on $\mathcal{H}_k = L^2(\mathrm{Gr}_k(P))$ generated by all left multiplication operators L_a, $a \in \mathrm{Gr}_k(P)$.

Remark 10.15.
Note that we also have the right multiplication operators R_a on $\mathcal{H}_k$, where $R_a(b) = b * a$ for $b \in \mathrm{Gr}_k(P)$. Put

$\Omega = 1_k =$  $\in P_{0,k}$

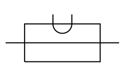
Then we have

$$\mathrm{tr}(a) = \langle a, \Omega \rangle = \langle a\Omega, \Omega \rangle,$$

Ω is cyclic and separating for M_k and we have

$$M_k = \mathrm{vN}(L_a \mid a \in \mathrm{Gr}_k(P)), \qquad M_k' = \mathrm{vN}(R_a \mid a \in \mathrm{Gr}_k(P)).$$

Definition 10.16.
We put

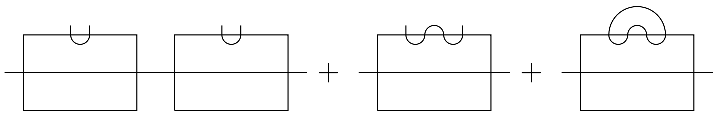
$U_k =$  $\in P_{2,k}$ (note $U_k^* = U_k$)

and define

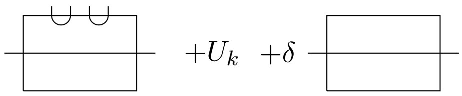
$$\mathcal{A}_k = \mathrm{alg}(U_k) \subset \mathrm{Gr}_k(P) \qquad \text{and} \qquad A_k = \mathrm{vN}(U_k) = \overline{\mathcal{A}_k}^w \subset M_k.$$

Remark 10.17.

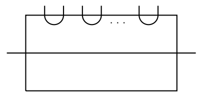
1) Note that $\mathcal{A}_k$ and A_k are commutative and

$U_k * U_k =$ 

i.e.

$U_k * U_k =$ 

and hence

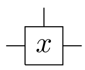
$\mathcal{A}_k = \text{span}\Big\{$  $\Big\}$

2) We will show $A' \cap M_k = AP_{0,k}$. (In particular, A is maximal commutative in M_k for $k = 0$.) From this we obtain then

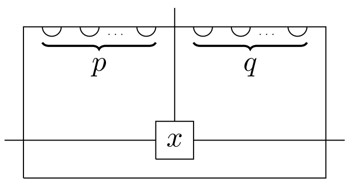
$$M_0' \cap M_k = P_{0,k} = P_k.$$

Notation 10.18.

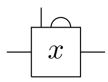
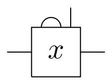
1) For

 $\in P_{n,k}$

we define

$x_{p,q} = \frac{1}{\delta^{\frac{p+q}{2}}}$  $\in P_{n+q+p,k}$

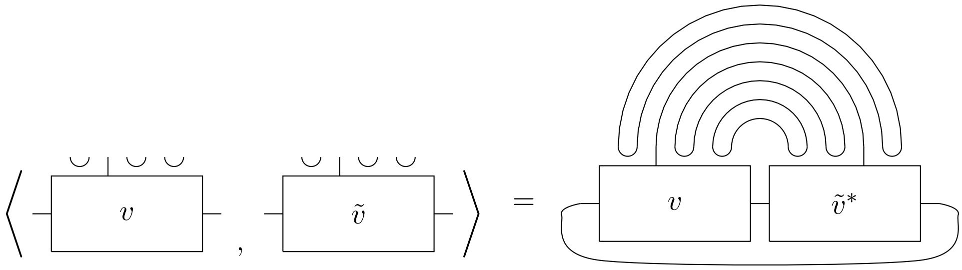
2) For $n \geq 0$, we define

$V_n := \text{span}\Big\{ x \in P_{n,k} \;\Big|\;$  $= 0 =$  $\Big\}$

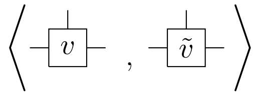
Lemma 10.19.
For $v \in V_n, \tilde{v} \in V_m$ and $n, m \geq 1$ we have

$$\langle v_{p,q}, \tilde{v}_{\tilde{p},\tilde{q}} \rangle = \begin{cases} \langle v, \tilde{v} \rangle & \text{if } p = \tilde{p},\ q = \tilde{q} \\ 0 & \text{otherwise} \end{cases}.$$

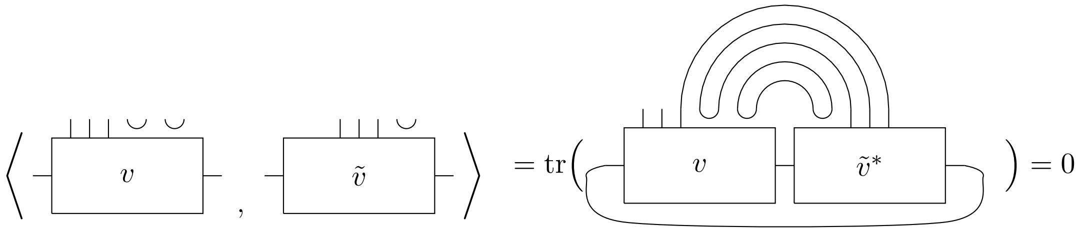
Proof. As an example for the first case we have



and thus

$\langle v_{1,3}, \tilde{v}_{1,3} \rangle =$ 

In the other case we have for example



q.e.d.

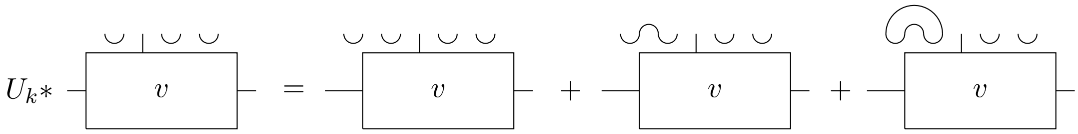
Theorem 10.20.
As an A-A bimodule we have the decomposition

$$L^2(M_k) = [P_{0,k} \otimes l^2(\mathbb{N})] \oplus [\mathcal{H} \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{N})],$$

where the second component is generated by elements $v_{p,q}$ (with v running over a basis of the V_n) and with the action

$$U_k * v_{p,q} = \begin{cases} \sqrt{\delta} v_{1,q} + v_{0,q} & p = 0 \\ \sqrt{\delta} v_{p+1,q} + v_{p,q} + \sqrt{\delta} v_{p-1,q} & p > 0 \end{cases}.$$

Proof. Consider for example



q.e.d.

Corollary 10.21.
We have $A' \cap M_k = AP_{0,k}$.

Proof. Assume $\xi \in M_k \subset L^2(M_k)$ is in A'. We have to show that ξ cannot have a component in $\mathcal{H} \otimes l^2(\mathbb{N}) \otimes l^2(\mathbb{N})$. Assume

$$\xi = \sum \alpha_{p,q} v_{p,q} \qquad \text{and} \qquad \xi * U_k = U_k * \xi.$$

Then we have

$$\sum \alpha_{p,q} \underbrace{U_k * v_{p,q}}_{v_{p+1,q}+v_{p,q}+v_{p-1,q}} = \sum \alpha_{p,q} \underbrace{v_{p,q} * U_k}_{v_{p,q+1}+v_{p,q}+v_{p,q-1}}.$$

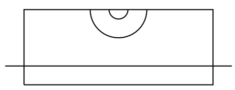
Hence, by the fact that $\xi \in L^2$, $\alpha_{p,q} = 0$ for all p, q. q.e.d.

Corollary 10.22.
If $\delta > 1$, then we have for each k that $M_0' \cap M_k = P_{0,k} = P_k$ (as algebras).

Proof. We have

$$M_0 \cap M_k \subset A' \cap M_k = AP_{0,k}$$

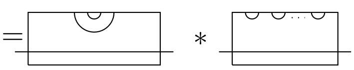
Assume $\xi \in AP_{0,k}$ is in $M_0' \cap M_k$, hence it commutes in particular with

$\alpha =$  $\in M_0 \subset M_k$

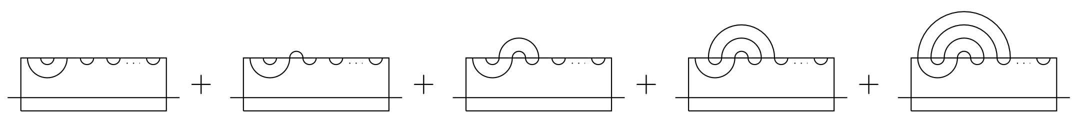
Write $\xi \in AP_{0,k}$ as an l^2 sum

$$\xi = \sum_{n=0}^{\infty} c_n * 1_{0,n}, \qquad \text{where } c_n \in P_{0,k}.$$

We have

$\alpha * 1_{0,n} =$ 

which gives



In the same way one gets $1_{0,n} * \alpha$, the only difference is that all contractions happen on the right. If we denote the first of the pictures above by λ_n and its right counterpart by ρ_n, then the difference between the two calculations is given by

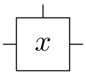
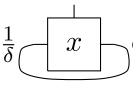
$$[\alpha, \xi] = \sum_{n=0}^{\infty} (c_n + \frac{1}{\sqrt{\delta}} c_{n+1})(\lambda_n - \varrho_n);$$

this implies that $c_{n+1} = -c_n\sqrt{\delta}$ for all $n \geq 1$, and hence, by the l^2-summability of the c_n, that $c_n = 0$ for all $n \geq 1$. q.e.d.

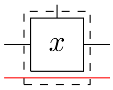
Remark 10.23.
It remains to see, that M_2 is given as the basic construction for $M_0 \subset M_1$, i.e. $M_2 = \langle M_1, e\rangle$, where e is the projection from M_1 onto M_0. The crucial property of e was

$$exe = E_{M_0}(x)e \qquad \text{for all } x \in M_1.$$

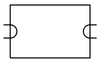
Let us check that this is the case. We have for

$x =$  $\in M_1$ that $E_{M_0}(x) = \frac{1}{\delta}$  $\in M_0$

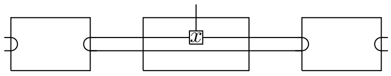
and $x \in M_1 \subset M_2$ is given in M_2 by adding a horizontal string, i.e.

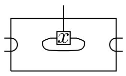
$x =$  $\in M_2$

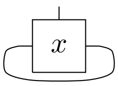
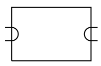
We put

$e = \frac{1}{\delta}$  $\in M_2$

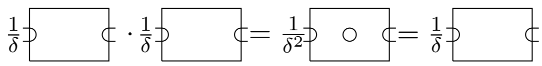
then we have

$exe = \frac{1}{\delta^2}$ 

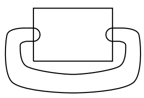
$= \frac{1}{\delta^2}$ 

$= \frac{1}{\delta}$  $\cdot \frac{1}{\delta}$ 

Note that e is a projection



and that

$\mathrm{tr}(e) = \frac{1}{\delta^3}$  $= \frac{1}{\delta^2}$

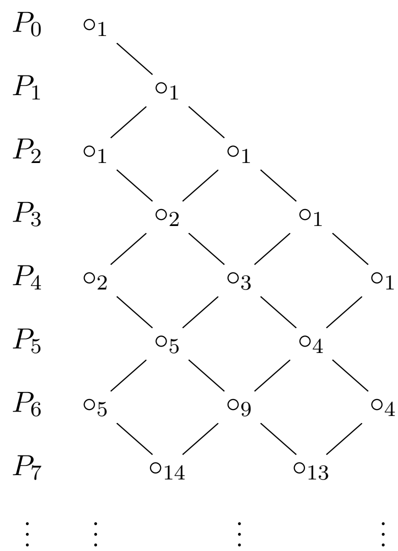
11 The Search for Subfactors With Small Index

The standard invariant for a subfactor $M_0 \in M_1$ is given by

$$\begin{array}{ccccccc} \mathbb{C} = M_0' \cap M_0 & \subset & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & M_0' \cap M_3 \quad \dots \\ & & \cup & & \cup & & \cup \\ & & \mathbb{C} = M_1' \cap M_1 & \subset & M_1' \cap M_2 & \subset & M_1' \cap M_3 \quad \dots \end{array}$$

This is the inclusion of finite dimensional C*-algebras, equipped with a consistent trace.

Part of the information is contained in the Bratteli-diagram, say for the first row (shown for TL in the following example)



The part on the $(n+1)$-th level is the reflection of the $(n-1)$-th level plus "new stuff"; and the new stuff on the $(n-1)$-th level is only connected to new stuff on the n-th level.

Definition 11.1.
The *principal graph* of our subfactor $M_0 \subset M_1$ is the graph Γ consisting of the new stuff; this is a bipartite graph. If the principal graph is finite, then the subfactor $M_0 \subset M_1$ has *finite depth*.

Remark 11.2.

1) Note that we have two principal graphs for a subfactor $M_0 \subset M_1$
   a) Γ for the inclusion $\{M_0' \cap M_k\}_k$,
   b) Γ' for the inclusion $\{M_1' \cap M_k\}_k$.
2) One can show
   a) The graph Γ is finite if and only if Γ' is finite.

b) The graphs Γ and Γ' do not need to be the same, but their "radius" differ at most by one.

c) If $\|\Gamma\|$ denotes the operator norm of the adjacency matrix of Γ (i.e., the largest eigenvalue, when Γ is finite), then we have in general

$$[M_1 : M_0] \geq \|\Gamma\|^2,$$

and with equality if Γ is finite. Actually, the index is equal to the norm squared of the principal graph precisely when $N \subset M$, hyperfinite II_1 factors, is amenable in the sense of Popa. Finite depth is a special case of this.

d) If $M_0 \subset M_1 \cong \mathcal{R}$ is amenable (for example, has finite depth) then the standard invariant determines the subfactors up to isomorphism. In this case the subfactor can be realized as the GNS construction for the embedding of the Jones tower. Hence for amenable sufactors classifying subfactors corresponds to classifying planar algebras. Otherwise these are different classification problems. (We know something about planar algebras, but non-amenable subfactors are out of reach for the moment).

Example 11.3.

1) For $[M_1 : M_0] < 4$ one has to find all Γ with $\|\Gamma\| < 2$. Those are all finite and can be classified as follows:

   a) the graphs A_n are given by

$A_n =$ 

$n = 1$

$$A_1 \hat{=} (0) \rightsquigarrow \|A_1\| = 0.$$

$n = 2$

$$A_2 \hat{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \|A_2\| = 1, \qquad \text{index} = 1^2 = 1$$

$n = 3$

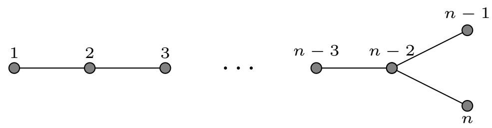
$$A_3 \hat{=} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \|A_3\| = \sqrt{2}, \qquad \text{index} = \sqrt{2}^2 = 2$$

In general

$$\|A_n\| = 2\cos\left(\frac{\pi}{n+1}\right),$$

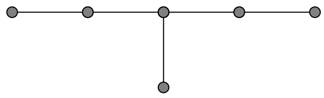
corresponding to the possible indices $4\cos^2\left(\frac{\pi}{n+1}\right)$.

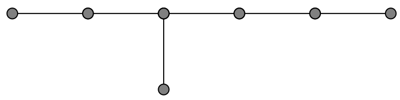
b) For $n \geq 4$, the graphs D_n are given by

$D_n =$ 

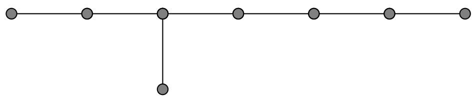
with $\|D_n\| = 2\cos\left(\frac{\pi}{2n-2}\right)$.

c) We have

$E_6 =$ 

$E_7 =$ 

and

$E_8 =$ 

with $\|E_6\| = 2\cos\left(\frac{\pi}{12}\right)$, $\|E_7\| = 2\cos\left(\frac{\pi}{18}\right)$ and $\|E_8\| = 2\cos\left(\frac{\pi}{30}\right)$.

In the 1990s one has classified all possible subfactor planar algebras which can have those graphs as principal graphs:

Principal graph	A_n	D_{2n+1}	D_{2n}	E_6	E_7	E_8
Realization	1	0	1	2	0	2

Since those graphs have finite depth, this gives all subfactors for the hyperfinite II_1 factor $\mathcal{R}$ with index < 4. For example in the case of index $4\cos^2\left(\frac{\pi}{30}\right)$ there are 4 different subfactors of $\mathcal{R}$; one with principal graph A_{29}, one with D_{16} and two different ones with E_8.

2) One can also classify all possible principal graphs with $\|\Gamma\| = 2$ and find all corresponding standard invariants. Not all of them are finite depth, but they are still amenable, hence determine in the hyperfinite case still the subfactor.

3) For index bigger than 4 things are getting complicated.

Remark 11.4. 1) Consider the infinite graph

$A_\infty =$ 

then

$$A_\infty \hat{=} \begin{bmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

has norm 2. This is the principal graph of the Temperley-Lieb planar algebra $TL(\delta)$. For $\delta = 2$ this is still amenable, hence there is one hyperfinite subfactor with A_∞ standard invariant and index 4. For each index > 4 there is at least one subfactor with A_∞ standard invariant. We do not know if it can be realized with hyperfinite subfactors, but it can with subfactors isomorphic to $L(\mathbb{F}_\infty)$ (by a result of Popa and Shlyakhtenko). For the hyperfinite case one knows that there is a subfactor of the hyperfinite II_1 factor with index 4.02642... (though this index is the norm squared of the graph E_{10}, the standard invariant of this subfactor is still A_∞) and it is actually claimed by Popa (announced in 1990, but no published proof yet) that this is the smallest possible index > 4 for an irreducible subfactor of the hyperfinite II_1 factor.
[Note that irreducibility is part of the data given by the standard invariant, hence every subfactor with A_∞ standard invariant is irreducible.]

2) For index > 4 one also has a standard invariant $A_{-\infty,\infty}$. This corresponds to Jones easy construction of a hyperfinite subfactor for every number > 4, see Example 4.8. But those are reducible. Finding irreducible subfactors with index >4 is highly non-trivial.

3) In the range $4 <$index< 5, there are only the following possibilities for irreducible subfactors of the hyperfinite II_1 factor.

   a) $TL_n(\delta)$ with principal graph A_∞. But it is not known for which indices one actually has hyperfinite subfactors. As mentioned above, there is one at 4.02642..., but not much more is known. It could be that the values are dense starting at some point between 4 and 5.

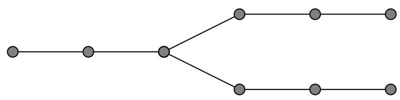
   b) 10 explicitly known planar algebras, each of them having finite depth and hence there is exactly one hyperfinite subfactor corresponding to each of them.

4) There is some hope for similar classification up to somewhere between 5 and 6.

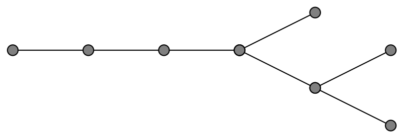
5) At the latest at index 6 things are getting pretty wild and all hope is gone. One knows the subfactor planar algebras up to index $3 + \sqrt{6}$, but not all the subfactors.

6) The idea to find subfactors with small index is to look for graphs with small norms and then to find (or rule out) planar algebras which have those as principal graphs. All this is quite non-trivial (and requires a lot of consistency checks by computer).

7) An interesting subfactor is the Haagerup subfactor with principal graphs



and



with index

$$\|\Gamma\|^2 = \frac{5+\sqrt{13}}{2} \approx 4.302.$$

This is actually getting quite complicated; for getting this sorted out one should consult the nice survey article “The classification of subfactors of index at most 5” (by V.F.R. Jones, S. Morrison, N. Snyder, Bulletin of the AMS 51, 2014) or just ask Dietmar Bisch or Vaughan Jones.

12 Exercises

Exercise 1.

(a) Let ϕ be a linear functional on some $B(\mathcal{H})$. Prove that the following statements are equivalent:

(i) There are $n \in \mathbb{N}$ and vectors $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$, such that

$$\phi(x) = \sum_{k=1}^{n} \langle x\xi_k, \eta_k \rangle \qquad \text{for all } x \in B(\mathcal{H}).$$

(ii) ϕ is continuous with respect to the weak operator topology.

(iii) ϕ is continuous with respect to the strong operator topology.

Show that the equivalence of (ii) and (iii) still holds for a linear functional ϕ defined on any von Neumann algebra $M \subset B(\mathcal{H})$.

In fact, for any linear functional ϕ on a von Neumann algebra $M \subset B(\mathcal{H})$, the following statements are equivalent (see Blackadar, Theorem III.2.1.4):

(i) There are sequences $(\xi_k)_{k\in\mathbb{N}}$, $(\eta_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ with $\sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty$ and $\sum_{k=1}^{\infty} \|\eta_k\|^2 < \infty$, such that $\phi(x) = \sum_{k=1}^{\infty} \langle x\xi_k, \eta_k \rangle$ for all $x \in M$.

(ii) ϕ, restricted to the unit ball of M, is continuous with respect to the weak operator topology.

(iii) ϕ, restricted to the unit ball of M, is continuous with respect to the strong operator topology.

(iv) ϕ is normal.

If ϕ is a state, these are also equivalent to

(v) There is an orthogonal sequence $(\xi_k)_{k\in\mathbb{N}}$ of vectors in $\mathcal{H}$ with $\sum_{k=1}^{\infty} \|\xi_k\|^2 = 1$, such that $\phi(x) = \sum_{k=1}^{\infty} \langle x\xi_k, \xi_k \rangle$ for all $x \in M$.

(vi) ϕ is completely additive, i.e., whenever $(p_i)_{i\in I}$ is a family of mutually orthogonal projections in M, then $\phi\big(\sum_{i\in I} p_i\big) = \sum_{i\in I} \phi(p_i)$.

(b) Let M be a finite factor and let τ be the unique norm-continuous trace on M (see Theorem 2.14). As usually, we denote by $L^2(M) = L^2(M, \tau)$ the complex Hilbert space obtained by completion of M with respect to the inner product $\langle \cdot, \cdot \rangle$ induced by τ, i.e. $\langle x, y \rangle := \tau(xy^*)$ for all $x, y \in M$. The corresponding norm on $L^2(M)$ will be denoted by $\| \cdot \|_2$.

Consider now the unit ball $B := \{x \in M | \; \|x\| \leq 1\}$ with respect to the operator norm $\| \cdot \|$ on M. Show that B, endowed with the metric induced by $\| \cdot \|_2$, is a complete metric space and that the topology on B induced by $\| \cdot \|_2$ is the same as the strong operator topology.

Exercise 2. Let $M \subset B(\mathcal{H})$ be a type II_1-factor with trace τ, acting on some Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where M possesses a cyclic and separating vector Ω such that $\tau(x) = \langle x\Omega, \Omega \rangle$ for all $x \in M$. Denote by M' the commutant of M and let $J : \mathcal{H} \to \mathcal{H}$ be the *antilinear unitary involution* determined by $J(x\Omega) = x^*\Omega$ for all $x \in M$. Prove the following statements:

(a) For all $\xi, \eta \in \mathcal{H}$, we have $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$.

(b) For all $x, a \in M$, it holds true that $JxJ(a\Omega) = ax^*\Omega$.

(c) If $x \in M'$ is given, we have $Jx\Omega = x^*\Omega$.

Deduce finally that $JMJ = M'$.

Hint: Switch the roles of M and M'. What does (c) tell us about this case?

Exercise 3. Fix any integer $m \in \mathbb{N}$, $m \geq 2$. Consider the chain of inclusions

$$M_m(\mathbb{C}) \hookrightarrow M_{m^2}(\mathbb{C}) \hookrightarrow M_{m^3}(\mathbb{C}) \hookrightarrow \ldots \hookrightarrow M_{m^n}(\mathbb{C}) \hookrightarrow M_{m^{n+1}}(\mathbb{C}) \hookrightarrow \ldots,$$

which are given by

$$M_{m^n}(\mathbb{C}) \hookrightarrow M_{m^{n+1}}(\mathbb{C}), \quad B \mapsto \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}.$$

(a) Justify that its union $M^{(m)} := \bigcup_{n \in \mathbb{N}} M_{m^n}(\mathbb{C})$ is a complex unital algebra and show that there exists a (well-defined!) tracial linear functional $\tau^{(m)} : M^{(m)} \to \mathbb{C}$ such that $\tau^{(m)}(B) = \mathrm{tr}_{m^n}(B)$ holds for any $B \in M_{m^n}(\mathbb{C})$. Recall that tr_{m^n} denotes the normalized trace on $M_{m^n}(\mathbb{C})$.

(b) Denote by $\mathcal{H}^{(m)}$ the Hilbert space which is obtained by completion of $M^{(m)}$ with respect to the inner product given by $\langle A, B \rangle_{(m)} = \tau^{(m)}(AB^*)$. Prove that each $B \in M^{(m)}$ induces a bounded linear operator on $\mathcal{H}^{(m)}$, i.e., we can view $M^{(m)} \subset B(\mathcal{H}^{(m)})$.

(c) Consider the von Neumann algebra $\mathcal{R} := \overline{M^{(m)}}^{\mathrm{sot}} \subset B(\mathcal{H}^{(m)})$. Show that there exists a unique normal tracial state τ on $\mathcal{R}$.

(d) Prove that $\mathcal{R}$ is a type II_1-factor.
Hint: Since the center $Z(\mathcal{R}) := \mathcal{R} \cap \mathcal{R}'$ of $\mathcal{R}$ is generated by its positive elements, factoriality follows as soon as we have shown that any positive $z \in Z(\mathcal{R})$ is a positive multiple of 1. For doing so, use the result obtained in (c).

It is a non-trivial result of Murray and von Neumann that $\mathcal{R}$ does not depend on the special choice of m. In fact, the obtained von Neumann algebra $\mathcal{R}$ is the hyperfinite II_1-factor. To see that $\mathcal{R}$ is isomorphic to $L(S_\infty)$, as the hyperfinite II_1-factor was introduced in the lecture, is again a non-trivial result of Murray and von Neumann.

Exercise 4. Let $M \subset B(\mathcal{H})$ be von Neumann algebra on some Hilbert space $\mathcal{H}$ and let $p \in M$ be a non-zero projection. Prove the following statements:

(a) We have $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on the Hilbert space $p\mathcal{H} = \mathrm{ran}(p)$. Thus pMp and $M'p$ are both von Neumann algebras on $p\mathcal{H}$.

Hint: First show that $(pM')' = pMp$ holds. Conclude by proving that any unitary $u \in (pMp)'$ can be extended to an isometry $\tilde{u} : \mathcal{K} \to \mathcal{K}$ on the Hilbert space $\mathcal{K} := \overline{Mp\mathcal{H}} \subset \mathcal{H}$, such that $\tilde{u}q \in M'$ holds for q being the orthogonal projection from $\mathcal{H}$ onto $\mathcal{K}$. For this purpose, check that $q \in Z(M)$.

(b) If M is a factor, then pMp and pM' are both factors on $p\mathcal{H}$. Moreover, the map

$$\Phi : \; M' \to M'p, \; x \mapsto xp$$

is a weakly continuous $*$-algebra isomorphism.

Hint: Use the general fact (which was proven in (a)) that the orthogonal projection q from $\mathcal{H}$ onto $\mathcal{K} = \overline{Mp\mathcal{H}}$ belongs to $Z(M)$.

(c) If M is a factor and if $x \in M$ and $y \in M'$ are given, then $xy = 0$ implies that $x = 0$ or $y = 0$.

(d) If M is a factor, then $M \cup M'$ generates $B(\mathcal{H})$ as a von Neumann algebra.

(e) If M is a type II_1-factor, then $pMp \subset B(p\mathcal{H})$ is also a type II_1-factor.

Exercise 5. Let M be a type II_1-factor and denote by τ_M its canonical trace. Prove the following properties of the coupling constant:

(a) If $(\mathcal{H}_i)_{i\in I}$ is a family of M-modules over a countable index set I, we have that

$$\dim_M\Big(\bigoplus_{i\in I}\mathcal{H}_i\Big) = \sum_{i\in I}\dim_M(\mathcal{H}_i).$$

(b) If $\mathcal{H}$ is an M-module and $p \in M$ a projection, then it holds true that

$$\dim_{pMp}(p\mathcal{H}) = \frac{1}{\tau_M(p)}\,\dim_M(\mathcal{H}).$$

(c) Consider the commutant M' of M with respect to its standard representation on $L^2(M)$. If $q \in M'$ is a projection, we have that

$$\dim_M(qL^2(M)) = \tau_{M'}(q).$$

(d) Assume that $\mathcal{H}$ is an M-module for which M' is also a type II_1-factor. We denote the canonical trace of M' by $\tau_{M'}$. For any $p \in M'$, it holds true that

$$\dim_{Mp}(p\mathcal{H}) = \tau_{M'}(p)\dim_M(\mathcal{H}).$$

Exercise 6. Consider the type I_n-factor $M = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and denote by tr_n its normalized trace.

(a) Discuss the statements (a) – (d) of Exercise 4 in each of the two cases

$$\mathcal{H} = \mathbb{C}^n \qquad \text{and} \qquad \mathcal{H} = L^2(M)$$

(b) It is known that each representation of M on a finite dimensional Hilbert space $\mathcal{H}$ is unitarily equivalent (in analogy to Definition 3.15) to a representation of the form

$$M \to B(\mathbb{C}^n \otimes \mathbb{C}^k) = M_n(\mathbb{C}) \otimes M_k(\mathbb{C}), \ x \mapsto x \otimes 1$$

for some $k \in \mathbb{N}_0$. In this case, we put

$$\dim_M(\mathcal{H}) := \frac{k}{n}.$$

What are the correct analogues of the properties (a) – (d) in Exercise 5?

Exercise 7. Let H and G be discrete i.c.c. groups, such that H is a subgroup of G. We denote by $[G : H]$ the group theoretic index of H in G, i.e. the number of (left or right) cosets of H in G. Recall that left and right cosets of H in G are of the form $gH = \{gh | \ h \in H\}$ and $Hg = \{hg | \ h \in H\}$ for $g \in G$, respectively, and that their number is always the same.

(a) Justify that $\ell^2(G)$ provides an $L(H)$-module and prove that its $L(H)$-dimension is given by

$$\dim_{L(H)}(\ell^2(G)) = [G : H].$$

(b) Consider the group factor $L(G)$ and denote by τ its canonical trace. Show that

$$L^2(L(G), \tau) \qquad \text{and} \qquad \ell^2(G)$$

are isomorphic as $L(G)$-modules.

(c) Show that $L(H)$ can be considered as a subfactor of $L(G)$ and deduce for the corresponding Jones index that

$$[L(G) : L(H)] = [G : H].$$

Exercise 8. Let $M \subseteq B(\mathcal{H})$ be a finite dimensional von Neumann algebra.

(a) Prove that pMp is a factor on $p\mathcal{H}$ for each minimal projection p in the center $Z(M)$.

(b) Show that the center $Z(M)$ is a finite dimensional abelian von Neumann algebra, which can be written as

$$Z(M) = \bigoplus_{i=1}^{k} (\mathbb{C}p_i),$$

where $p_1, \ldots, p_k$ denote the minimal projections in $Z(M)$.

(c) Deduce that there are $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}$, such that M is isomorphic to

$$\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}).$$

In fact, there are Hilbert spaces $\mathcal{K}_1, \dots, \mathcal{K}_k$ and a unitary $u : \bigoplus_{i=1}^{k}(\mathbb{C}^{n_i} \otimes \mathcal{K}_i) \to \mathcal{H}$, such that

$$u^* M u = \bigoplus_{i=1}^{k} (B(\mathbb{C}^{n_i}) \otimes 1).$$

Hint: You may use without giving a proof that for any type I_n-factor M on a Hilbert space $\mathcal{H}$, there exists a Hilbert space $\mathcal{K}$ and a unitary $u : \mathbb{C}^n \otimes \mathcal{K} \to \mathcal{H}$, such that $uMu^* = B(\mathbb{C}^n) \otimes 1$.

Exercise 9.

(a) Let M be a factor of type I_n. Prove that any subfactor N of M is of type I_m for some integer m dividing n. Moreover, show that all subfactors N of M of type I_m are uniquely determined, up to conjugation by unitaries in M, by the integer $k > 0$ such that pMp is a factor of type I_k for some minimal projection $p \in N$ and $mk = n$.

(b) Let $N \subseteq M$ be finite dimensional von Neumann algebras. Let $p_1, \dots, p_m$ be the minimal central projections of M and $q_1, \dots, q_n$ those of N. For each $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, $p_j q_i M q_i p_j$ yields a factor with subfactor $p_j q_i N$, to which we may associate an integer $k_{i,j}$ according to (a). We form the matrix

$$\Lambda = (k_{i,j})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}.$$

Compute Λ for $M = M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ and the subalgebra N of matrices of the form

$$\begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z \end{pmatrix} \oplus \begin{pmatrix} X & 0 \\ 0 & z \end{pmatrix} \qquad \text{with } z \in \mathbb{C} \text{ and } X \in M_2(\mathbb{C}).$$

Often, the matrix

$$\Lambda = (k_{i,j})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$$

is represented by a bipartite graph $G = (V, E)$ on the vertex set $V = P \dot{\cup} Q$ with $P = \{p_1, \dots, p_m\}$ and $Q = \{q_1, \dots, q_n\}$, which has $k_{i,j}$ edges between q_i and p_j. The obtained graph G is called the *Bratteli diagram* for $N \subseteq M$.

(c) Show that $k_{i,j} = \mathrm{Tr}(p_j e_i)$ holds, if e_i is a minimal projection in the factor $q_i N$. Note that Tr denotes here the unnormalized trace on $p_j M p_j$, which is isomorphic $M_{m_j}(\mathbb{C})$ for some $m_j \in \mathbb{N}$.

Exercise 10. Let $p, q \in B(\mathcal{H})$ be orthogonal projections on a separable complex Hilbert space $\mathcal{H}$.

(a) Show that

$$\operatorname*{s\text{--}lim}_{n\to\infty} (pqp)^n = p \wedge q.$$

Hint: First show (for instance by using functional calculus) that the sequence $((pqp)^n)_{n=1}^\infty$ converges strongly to some projection $e \in B(\mathcal{H})$. Finally, in order to identify e as $p \wedge q$, consider expressions of the form $(pqp)^m q (pqp)^n$ for $m, n \in \mathbb{N}$.

(b) Deduce the formula stated in Remark 5.13 (3), i.e., prove that

$$\operatorname*{s\text{--}lim}_{n\to\infty} (pq)^n = p \wedge q.$$

(c) Discuss the statements (a) and (b) in the case $\mathcal{H} = \mathbb{C}^3$ for the projections

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q = u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} u^*, \quad \text{where} \quad u = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

for some $0 < \theta < \pi$.

Exercise 11. Let $(S_n)_{n=0}^\infty$ be the sequence of *Chebyshev polynomials of the second kind*, which are recursively defined by $S_0(x) = 1$, $S_1(x) = x$ and

$$x S_n(x) = S_{n+1}(x) + S_{n-1}(x) \qquad \text{for all } n \geq 1.$$

Prove the following statements:

(a) For all $n \geq 0$ and all $0 < \theta < \pi$, it holds true that

$$S_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

(b) We have for all $n, m \geq 0$ that

$$\int_{-2}^{2} S_n(x) S_m(x) \, \frac{1}{2\pi} \sqrt{4 - x^2} \, dx = \delta_{n,m}.$$

(c) For all $x \in [-2, 2]$ and all $z \in \mathbb{C}$ with $|z| < 1$, we have that

$$\frac{1}{1 - xz + z^2} = \sum_{n=0}^{\infty} S_n(x) z^n.$$

(d) For $x, y \in [-2, 2]$ and all $n \geq 0$, it holds true that

$$\frac{S_n(x) - S_n(y)}{x - y} = \sum_{k=1}^{n} S_{k-1}(x) S_{n-k}(y).$$

Given any finite dimensional von Neumann algebra M, we know from Exercise 8 (c), that

$$M \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C})$$

for some $l \in \mathbb{N}$ and $m_1, \ldots, m_l \in \mathbb{N}$. For any choice of $\vec{t} = (t_1, \ldots, t_l)^T \in \mathbb{R}_+^l$, where $\mathbb{R}_+ := (0, \infty)$, we can thus introduce a faithful trace τ on M by

$$\tau := (t_1 \operatorname{Tr}_{m_1}) \oplus \cdots \oplus (t_l \operatorname{Tr}_{m_l}).$$

In fact, it is easy to see that any faithful trace τ on M arises in this way, and in this case the corresponding vector $\vec{t}$ is called the *trace vector of* τ. Obviously, the trace τ is normalized (i.e. $\tau(1) = 1$), if and only if $t_1 m_1 + \cdots + t_l m_l = 1$ holds.

Exercise 12.

(a) In Exercise 9 (b) we have constructed for any inclusion $N \subseteq M$ of finite dimensional von Neumann algebras a matrix Λ_N^M. Consider now finite dimensional von Neumann algebras $N \subseteq M \subseteq P$. Prove that the matrices corresponding to these inclusions satisfy the relation

$$\Lambda_N^P = \Lambda_N^M \Lambda_M^P.$$

(b) Take finite dimensional von Neumann algebras $N \subseteq M$, satisfying

$$N \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \qquad \text{and} \qquad M \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C}),$$

with the matrix

$$\Lambda_N^M = (\Lambda_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,l}}$$

constructed according to Exercise 9 (b). Moreover, let τ_N and τ_M be a faithful tracial states on N and M, respectively, with corresponding trace vectors $\vec{s} = (s_1, \ldots, s_k)^T$ and $\vec{t} = (t_1, \ldots, t_l)^T$. Prove that $\tau_M|_N = \tau_N$ if and only if $\Lambda_N^M \vec{t} = \vec{s}$.

It can be shown that the basic construction works equally well in the non-factor case. More precisely, in the situation of the previous exercise and under the assumption that $\tau_M|_N = \tau_N$ holds, we can find a projection $e_N \in B(L^2(M, \tau_M))$, such that $e_N(x\Omega) = E_N(x)\Omega$ holds for all $x \in M$, where $\Omega = \hat{1} \in L^2(M, \tau_M)$ and E_N denotes the conditional expectation from M to N as in Theorem 5.2. We consider then the von Neumann algebra $\langle M, e_N \rangle \subseteq B(L^2(M, \tau_M))$ generated by M and e_N.

Lemma (Jones, 1983)**.**
Let $p_1, \ldots, p_k$ be the minimal central projections of N. Then

(i) $Jp_1J, \ldots, Jp_kJ$ are the minimal central projections of $\langle M, e_N \rangle$,

(ii) $\Lambda_M^{\langle M, e_N \rangle} = (\Lambda_N^M)^T$ (with the obvious identification of the indices $p_i \leftrightarrow Jp_iJ$),

(iii) $e_N Jp_iJ = e_N p_i$,

(iv) $x \mapsto e_N x Jp_iJ$ is an isomorphism from p_iN onto $(e_N Jp_iJ)\langle M, e_N \rangle(e_N Jp_iJ)$.

Exercise 13. Consider the finite dimensional von Neumann algebras $N \subseteq M$ given by

$$\begin{aligned} \mathbb{C} \oplus \mathbb{C} &\xrightarrow{\cong} N \quad \subseteq \quad M := M_2(\mathbb{C}) \\ z_1 \oplus z_2 &\mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \end{aligned}$$

We endow M with the usual trace $\tau_M = \mathrm{tr}_2$ and N with the restriction $\tau_N = \tau_M|_N$. Compute $e_N \in B(L^2(M, \tau_M))$ and check explicitly the validity of the statements made in the lemma above.

Following Jones, we call a faithful tracial state τ on $M_1 = \langle M, e_N \rangle$ a (λ, P)-*trace*, for $\lambda > 0$ and a subalgebra P of M_1, if τ extends τ_M and $\tau(e_N x) = \lambda \tau_M(x)$ holds for all $x \in P$.

Theorem (Jones, 1983)**.**
Given $\lambda > 0$, there exists a (λ, M)-trace on M_1, if and only if

$$\Lambda^T \Lambda \vec{t} = \lambda^{-1} \vec{t} \qquad \text{and} \qquad \Lambda \Lambda^T \vec{s} = \lambda^{-1} \vec{s}, \qquad \text{where } \Lambda = \Lambda_N^M. \tag{12.1}$$

Exercise 14.

(a) Show that a (λ, N)-trace on M_1 is also a (λ, M)-trace on M_1.

(b) Prove the theorem above of Jones.

(c) Show that if condition (12.1) is satisfied for finite dimensional von Neumann algebras $N \subseteq M$, endowed with traces such that $\tau_M|_N = \tau_N$ holds, then the basic construction can be iterated in the sense that there is a (λ, M)-trace on $M_1 = \langle M, e_N \rangle$, a (λ, M_1)-trace on $M_2 = \langle M_1, e_M \rangle$, and so on.

It was observed by Jones that the projections appearing in the Jones tower

$$N \subseteq M \overset{e_1 = e_N}{\subseteq} M_1 \overset{e_2 = e_M}{\subseteq} M_2 \overset{e_3 = e_{M_1}}{\subseteq} M_3 \overset{e_4 = e_{M_2}}{\subseteq} \dots$$

constructed according to part (c) of the previous exercise can be used to build a subfactor $P_\lambda \subseteq P$ with Jones index $[P : P_\lambda] = \lambda^{-1}$. In fact, it can be shown that P is isomorphic to the hyperfinite II_1-factor.

Exercise 15. Let $n \in \mathbb{N}$ with $n \geq 2$ be fixed. Consider the symmetric matrix $\Lambda = (\Lambda_{ij})_{i,j=1}^n$ defined by

$$\Lambda_{ij} := \begin{cases} 1, & \text{if } |i - j| = 1 \\ 0, & \text{else} \end{cases} \qquad \text{for } i, j = 1, \dots, n,$$

i.e., we have

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

(a) Prove that the eigenvalues of Λ are precisely the zeros of the n-th Chebyshev polynomial S_n of the second kind (cf. Exercise 11), i.e.

$$\Big\{2\cos\Big(\frac{k\pi}{n+1}\Big)\Big|\, k = 1, \dots, n\Big\},$$

where an eigenvector corresponding to the eigenvalue $\lambda_k = 2\cos\big(\frac{k\pi}{n+1}\big)$ is given by

$$\vec{t}_k = \Big(\sin\Big(\frac{k\pi}{n+1}\Big), \sin\Big(\frac{2k\pi}{n+1}\Big), \dots, \sin\Big(\frac{nk\pi}{n+1}\Big)\Big)^T.$$

(b) Deduce that all values in

$$\Big\{4\cos^2\Big(\frac{\pi}{n+1}\Big)\Big|\, n \geq 2\Big\}$$

show up as the Jones index for some subfactor of the hyperfinite II_1-factor.

It is worth to point out that (12.1) gives an interesting connection to the famous Perron-Frobenius Theorem. More precisely, the existence of a positive eigenvector $\vec{t}$ for the matrix $P = \Lambda\Lambda^T$ (or analogously $\vec{s}$ for $P = \Lambda^T\Lambda$) implies that the corresponding eigenvalue λ^{-1} determines its norm by $\|\Lambda\|^2 = \|P\| = \lambda^{-1}$ and hence the Jones index of the constructed subfactor $P_\lambda \subseteq P$, i.e. $\|\Lambda\|^2 = [P : P_\lambda]$. However, for this purpose, we do not need the Perron-Frobenius Theorem in full generality. Hence, a more specialized proof (which nevertheless follows ideas of the general proof) is more appropriate.

Exercise 16. Let a real matrix $P = (P_{ij})_{i,j=1}^n \in M_n(\mathbb{R})$ be given, which is both symmetric (i.e. $P^T = P$) and non-negative (i.e. $P_{ij} \geq 0$ for all $i, j = 1, \dots, n$). Moreover, assume that there exists a real eigenvector $y = (y_1, \dots, y_n)^T$ of P, which satisfies $y_1, \dots, y_n > 0$, with corresponding eigenvalue $\lambda \geq 0$.

(a) On the set

$$\Gamma_n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n |\ x_1, \dots, x_n > 0\}$$

consider the function

$$L :\ \Gamma_n \to [0, \infty),\ x \mapsto \max\{s \geq 0 |\ sx \leq Px\},$$

where $x \leq x'$ for real vectors $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n)$ means that $x_i \leq x'_i$ holds for all $i = 1, \ldots, n$. Prove that

$$\sup_{x \in \Gamma_n} L(x) = \lambda = L(y).$$

Hint: Consider the inner product of $\langle x, y \rangle$ for $x \in \Gamma_n$ and check that we always have $\langle x, y \rangle > 0$ in this case.

(b) Deduce that $\|P\| = \lambda$.

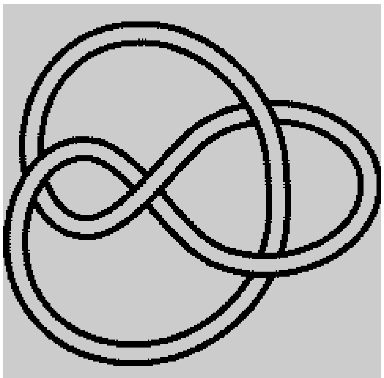
Hint: Note that if $\lambda_1, \ldots, \lambda_n$ are the ordered eigenvalues of any symmetric real matrix P, listed according to their multiplicity, then $\|P\| = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Exercise 17.

(a) Find a braid b whose closure $\hat{b}$ yields the following link and compute its Jones polynomial $V_{\hat{b}}(t)$.

Hint: Note that there are actually *two different* Jones polynomials related to the picture above, depending on the choice of an orientation on both of its components, since this will change the corresponding element in the braid group.

(b) Find a braid b whose closure $\hat{b}$ yields the following knot and compute its Jones polynomial $V_{\hat{b}}(t)$.



Hint: Choose any point P in the plane, which does not belong to the given projection of the knot, and fix an orientation of the knot. Try to deform the knot until its orientation on any subarc goes in mathematical positive sense around P. Decompose the obtained projection of the knot in sectors around P, such that each sector contains at most one crossing of the knot.

Exercise 18 (*von Neumann mean ergodic theorem*)**.** Let $\mathcal{H}$ be a separable complex Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n \xi = \pi(\xi) \tag{12.2}$$

holds for any $\xi \in \mathcal{H}$, where π denotes the orthogonal projection from $\mathcal{H}$ onto the closed subspace $\mathcal{H}^U$ of all U-invariant vectors in $\mathcal{H}$, i.e. $\mathcal{H}^U := \{\xi \in \mathcal{H} |\ U\xi = \xi\}$.

Hint: Consider the (possibly non-closed) subspace $\mathcal{W} := \{U\xi - \xi |\ \xi \in \mathcal{H}\}$ of $\mathcal{H}$ and show that

(a) $\mathcal{W}$ and $\mathcal{H}^U$ are orthogonal,

(b) formula (12.2) holds separately on $\mathcal{H}^U$ and $\mathcal{W}$ (and hence also on $\mathcal{H}^U + \mathcal{W}$),

(c) formula (12.2) holds on $\overline{\mathcal{H}^U + \mathcal{W}}$,

(d) we have $\overline{\mathcal{H}^U + \mathcal{W}} = \mathcal{H}$.