Free and cyclic differential calculus and characterizations of gradients

Tobias Mai

(joint work with Roland Speicher)

Saarland University

Probabilistic Operator Algebra Seminar

University of California, Berkeley

April 20, 2020

Free and cyclic differential calculus

Free and cyclic differential calculus

Let $\mathbb{C}\langle \underline{x} \rangle = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the algebra of noncommutative polynomials in the n formal variables $\underline{x} = (x_1, \dots, x_n)$; each $p \in \mathbb{C}\langle \underline{x} \rangle$ is of the form

$$p = \sum_{k=0}^{d} \sum_{1 \leq i_1, \dots, i_k \leq n} a_{i_1, \dots, i_k} x_{i_1} \cdots x_{i_k}.$$

Free and cyclic differential calculus

Let $\mathbb{C}\langle \underline{x} \rangle = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the algebra of noncommutative polynomials in the n formal variables $\underline{x} = (x_1, \dots, x_n)$; each $p \in \mathbb{C}\langle \underline{x} \rangle$ is of the form

$$p = \sum_{k=0}^{d} \sum_{1 \leq i_1, \dots, i_k \leq n} a_{i_1, \dots, i_k} x_{i_1} \cdots x_{i_k}.$$

Definition

- The free derivatives are the unique derivations

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \quad \text{satisfying} \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1.$$

Free and cyclic differential calculus

Let $\mathbb{C}\langle \underline{x} \rangle = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the algebra of noncommutative polynomials in the n formal variables $\underline{x} = (x_1, \dots, x_n)$; each $p \in \mathbb{C}\langle \underline{x} \rangle$ is of the form

$$p = \sum_{k=0}^{d} \sum_{1 \leq i_1, \dots, i_k \leq n} a_{i_1, \dots, i_k} x_{i_1} \cdots x_{i_k}.$$

Definition

- The free derivatives are the unique derivations

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \quad \text{satisfying} \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1.$$

- The cyclic derivatives are the linear mappings

$$\mathcal{D}_1, \dots, \mathcal{D}_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \quad \text{defined by} \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j,$$

Free and cyclic differential calculus

Let $\mathbb{C}\langle \underline{x} \rangle = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the algebra of noncommutative polynomials in the n formal variables $\underline{x} = (x_1, \dots, x_n)$; each $p \in \mathbb{C}\langle \underline{x} \rangle$ is of the form

$$p = \sum_{k=0}^{d} \sum_{1 \leq i_1, \dots, i_k \leq n} a_{i_1, \dots, i_k} x_{i_1} \cdots x_{i_k}.$$

Definition

- The free derivatives are the unique derivations

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \quad \text{satisfying} \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1.$$

- The cyclic derivatives are the linear mappings

$$\mathcal{D}_1, \dots, \mathcal{D}_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \quad \text{defined by} \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j,$$

where

- $\sigma : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle, \quad p_1 \otimes p_2 \mapsto p_2 \otimes p_1$ (flip)

Free and cyclic differential calculus

Let $\mathbb{C}\langle \underline{x} \rangle = \mathbb{C}\langle x_1, \dots, x_n \rangle$ be the algebra of noncommutative polynomials in the n formal variables $\underline{x} = (x_1, \dots, x_n)$; each $p \in \mathbb{C}\langle \underline{x} \rangle$ is of the form

$$p = \sum_{k=0}^{d} \sum_{1 \leq i_1, \dots, i_k \leq n} a_{i_1, \dots, i_k} x_{i_1} \cdots x_{i_k}.$$

Definition

- The free derivatives are the unique derivations

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \quad \text{satisfying} \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1.$$

- The cyclic derivatives are the linear mappings

$$\mathcal{D}_1, \dots, \mathcal{D}_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \quad \text{defined by} \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j,$$

where
  - $\sigma : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle, \quad p_1 \otimes p_2 \mapsto p_2 \otimes p_1$ (flip)
  - $\mu : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad p_1 \otimes p_2 \mapsto p_1 p_2$ (multiplication)

Free derivatives

Free derivatives

$\partial_1, \ldots, \partial_n : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ derivations, $\partial_i x_j = \delta_{i=j} 1 \otimes 1$

Free derivatives

$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ derivations, $\partial_i x_j = \delta_{i=j} 1 \otimes 1$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

Free derivatives

$$\partial_1, \dots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \text{ derivations, } \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1$$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Free derivatives

$$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \text{ derivations, } \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1$$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Example

Let $n = 2$ and consider $p = x_1 x_2 x_1 x_2$. Then

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,$$
$$\partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Free derivatives

$$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \text{ derivations}, \quad \partial_i x_j = \delta_{i=j} 1 \otimes 1$$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Example

Let $n = 2$ and consider $p = x_1 x_2 x_1 x_2$. Then

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,$$
$$\partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Free derivatives

$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ derivations, $\partial_i x_j = \delta_{i=j} 1 \otimes 1$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Example

Let $n = 2$ and consider $p = x_1 x_2 x_1 x_2$. Then

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,$$
$$\partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Free derivatives

$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ derivations, $\partial_i x_j = \delta_{i=j} 1 \otimes 1$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Example

Let $n = 2$ and consider $p = x_1 x_2 x_1 x_2$. Then

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,$$
$$\partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Free derivatives

$\partial_1, \ldots, \partial_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ derivations, $\partial_i x_j = \delta_{i=j} 1 \otimes 1$

- $\partial_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a derivation in the sense that

$$\partial_j(p_1 p_2) = \partial_j(p_1) \cdot p_2 + p_1 \cdot \partial_j(p_2) \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Note that $\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle$ is a $\mathbb{C}\langle \underline{x} \rangle$-bimodule with respect to the action

$$p_1 \cdot (q_1 \otimes q_2) \cdot p_2 := (p_1 q_1) \otimes (q_2 p_2).$$

Example

Let $n = 2$ and consider $p = x_1 x_2 x_1 x_2$. Then

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,$$
$$\partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Cyclic derivatives

Cyclic derivatives

$\mathcal{D}_1, \dots, \mathcal{D}_n : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ linear, $\ \mathcal{D}_j := \mu \circ \sigma \circ \partial_j$

Cyclic derivatives

$$\mathcal{D}_1, \dots, \mathcal{D}_n : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \text{ linear}, \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j$$

- $\mathcal{D}_j : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is *not* a derivation if $n \geq 2$, but satisfies

$$\mathcal{D}_j(p_1 p_2) = \tilde{\partial}_j(p_1) \sharp p_2 + \tilde{\partial}_j(p_2) \sharp p_1 \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

Cyclic derivatives

$$\mathcal{D}_1, \ldots, \mathcal{D}_n : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \text{ linear}, \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j$$

- $\mathcal{D}_j : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is *not* a derivation if $n \geq 2$, but satisfies

$$\mathcal{D}_j(p_1 p_2) = \tilde{\partial}_j(p_1) \sharp p_2 + \tilde{\partial}_j(p_2) \sharp p_1 \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Here, we use the notation $\tilde{\partial}_j := \sigma \circ \partial_j$ and the bilinear map

$$\sharp : \ (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle) \times \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad (p_1 \otimes p_2) \sharp q := p_1 q p_2.$$

Cyclic derivatives

$$\mathcal{D}_1, \dots, \mathcal{D}_n : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle \text{ linear}, \quad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j$$

- $\mathcal{D}_j : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is *not* a derivation if $n \geq 2$, but satisfies

$$\mathcal{D}_j(p_1 p_2) = \tilde{\partial}_j(p_1) \sharp p_2 + \tilde{\partial}_j(p_2) \sharp p_1 \qquad \text{for all } p_1, p_2 \in \mathbb{C}\langle \underline{x} \rangle.$$

- Here, we use the notation $\tilde{\partial}_j := \sigma \circ \partial_j$ and the bilinear map

$$\sharp : (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle) \times \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad (p_1 \otimes p_2) \sharp q := p_1 q p_2.$$

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. We already know that

$$\partial_1 p = 1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2 \qquad \text{and} \qquad \partial_2 p = x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1.$$

Hence, we get that

$$\mathcal{D}_1 p = 2\, x_2 x_1 x_2 \qquad \text{and} \qquad \mathcal{D}_2 p = 2\, x_1 x_2 x_1.$$

Why are these natural notions?

Why are these natural notions?

- For every $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$, we have the evaluation homomorphism

$$\mathrm{ev}_{\underline{X}} : \mathbb{C}\langle \underline{x} \rangle \to M_N(\mathbb{C}), \quad x_{i_1} x_{i_2} \cdots x_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}.$$

Why are these natural notions?

- For every $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$, we have the evaluation homomorphism

$$\mathrm{ev}_{\underline{X}}: \ \mathbb{C}\langle \underline{x} \rangle \to M_N(\mathbb{C}), \quad x_{i_1} x_{i_2} \cdots x_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}.$$

- Consequently, every $p \in \mathbb{C}\langle \underline{x} \rangle$ induces matrix functions

$$p: \ M_N(\mathbb{C})^n \to M_N(\mathbb{C}), \quad \underline{X} \mapsto p(\underline{X}) := \mathrm{ev}_{\underline{X}}(p).$$

Why are these natural notions?

- For every $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$, we have the evaluation homomorphism

$$\mathrm{ev}_{\underline{X}}:\ \mathbb{C}\langle \underline{x} \rangle \to M_N(\mathbb{C}), \quad x_{i_1} x_{i_2} \cdots x_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}.$$

- Consequently, every $p \in \mathbb{C}\langle \underline{x} \rangle$ induces matrix functions

$$p:\ M_N(\mathbb{C})^n \to M_N(\mathbb{C}), \quad \underline{X} \mapsto p(\underline{X}) := \mathrm{ev}_{\underline{X}}(p).$$

At each *point* $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$ and for every *direction* $\underline{H} = (H_1, \dots, H_n) \in M_N(\mathbb{C})^n$, we have for the *directional derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\underline{X} + t\underline{H})\Big|_{t=0} = \sum_{j=1}^{n} (\partial_j p)(\underline{X}) \sharp H_j,$$

Why are these natural notions?

- For every $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$, we have the evaluation homomorphism

$$\mathrm{ev}_{\underline{X}} :\ \mathbb{C}\langle \underline{x} \rangle \to M_N(\mathbb{C}), \quad x_{i_1} x_{i_2} \cdots x_{i_k} \mapsto X_{i_1} X_{i_2} \cdots X_{i_k}.$$

- Consequently, every $p \in \mathbb{C}\langle \underline{x} \rangle$ induces matrix functions

$$p :\ M_N(\mathbb{C})^n \to M_N(\mathbb{C}), \quad \underline{X} \mapsto p(\underline{X}) := \mathrm{ev}_{\underline{X}}(p).$$

At each *point* $\underline{X} = (X_1, \dots, X_n) \in M_N(\mathbb{C})^n$ and for every *direction* $\underline{H} = (H_1, \dots, H_n) \in M_N(\mathbb{C})^n$, we have for the *directional derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\underline{X} + t\underline{H})\Big|_{t=0} = \sum_{j=1}^{n} (\partial_j p)(\underline{X}) \sharp H_j,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{tr}_N \big(p(\underline{X} + t\underline{H})\big)\Big|_{t=0} = \sum_{j=1}^{n} \mathrm{tr}_N \big((\mathcal{D}_j p)(\underline{X}) H_j\big).$$

Can we recognize free and cyclic gradients?

Can we recognize free and cyclic gradients?

Question 1

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{q} = \mathcal{D}p \qquad \text{for} \qquad \mathcal{D}p := (\mathcal{D}_1 p, \dots, \mathcal{D}_n p) \ ?$$

Can we recognize free and cyclic gradients?

Question 1

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{q} = \mathcal{D}p \qquad \text{for} \qquad \mathcal{D}p := (\mathcal{D}_1 p, \dots, \mathcal{D}_n p) \ ?$$

Question 2

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{u} = \partial p \qquad \text{for} \qquad \partial p := (\partial_1 p, \dots, \partial_n p) \ ?$$

Can we recognize free and cyclic gradients?

Question 1

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle\underline{x}\rangle^n$ be given. When does there exist a $p \in \mathbb{C}\langle\underline{x}\rangle$ such that

$$\underline{q} = \mathcal{D}p \qquad \text{for} \qquad \mathcal{D}p := (\mathcal{D}_1 p, \dots, \mathcal{D}_n p) \ ?$$

Question 2

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle\underline{x}\rangle \otimes \mathbb{C}\langle\underline{x}\rangle)^n$ be given. When does there exist a $p \in \mathbb{C}\langle\underline{x}\rangle$ such that

$$\underline{u} = \partial p \qquad \text{for} \qquad \partial p := (\partial_1 p, \dots, \partial_n p) \ ?$$

Goal in both cases

Can we recognize free and cyclic gradients?

Question 1

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{q} = \mathcal{D}p \qquad \text{for} \qquad \mathcal{D}p := (\mathcal{D}_1 p, \dots, \mathcal{D}_n p) \ ?$$

Question 2

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{u} = \partial p \qquad \text{for} \qquad \partial p := (\partial_1 p, \dots, \partial_n p) \ ?$$

Goal in both cases

- Formulate criteria which completely characterize gradients.

Can we recognize free and cyclic gradients?

Question 1

Let $\underline{q} = (q_1, \ldots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{q} = \mathcal{D}p \qquad \text{for} \qquad \mathcal{D}p := (\mathcal{D}_1 p, \ldots, \mathcal{D}_n p) \ ?$$

Question 2

Let $\underline{u} = (u_1, \ldots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. When does there exist a $p \in \mathbb{C}\langle \underline{x} \rangle$ such that

$$\underline{u} = \partial p \qquad \text{for} \qquad \partial p := (\partial_1 p, \ldots, \partial_n p) \ ?$$

Goal in both cases

- Formulate criteria which completely characterize gradients.
- Provide some recipe to construct an antiderivative p.

Number operator and cyclic symmetrization operator

Number operator and cyclic symmetrization operator

Definition

- The number operator $N : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is defined by

$$N1 = 0 \qquad \text{and} \qquad N x_{i_1} \cdots x_{i_k} = k\, x_{i_1} \cdots x_{i_k}.$$

Number operator and cyclic symmetrization operator

Definition

- The number operator $N : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is defined by

$$N1 = 0 \qquad \text{and} \qquad N x_{i_1} \cdots x_{i_k} = k\, x_{i_1} \cdots x_{i_k}.$$

- The cyclic symmetrization operator $C : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is defined by

$$C1 = 0 \qquad \text{and} \qquad C x_{i_1} \cdots x_{i_k} = \sum_{p=1}^{k} x_{i_{p+1}} \cdots x_{i_k} x_{i_1} \cdots x_{i_p}.$$

Number operator and cyclic symmetrization operator

Definition

- The number operator $N : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is defined by

$$N1 = 0 \qquad \text{and} \qquad N x_{i_1} \cdots x_{i_k} = k \, x_{i_1} \cdots x_{i_k}.$$

- The cyclic symmetrization operator $C : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ is defined by

$$C1 = 0 \qquad \text{and} \qquad C x_{i_1} \cdots x_{i_k} = \sum_{p=1}^{k} x_{i_{p+1}} \cdots x_{i_k} x_{i_1} \cdots x_{i_p}.$$

Put $\mathbb{C}^{(k)}\langle \underline{x} \rangle := \operatorname{span}_{\mathbb{C}}\{x_{i_1} \cdots x_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$. With respect to

$$\mathbb{C}\langle \underline{x} \rangle = \bigoplus_{k \geq 0} \mathbb{C}^{(k)}\langle \underline{x} \rangle \qquad \text{and} \qquad \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle = \bigoplus_{k,l \geq 0} \mathbb{C}^{(k)}\langle \underline{x} \rangle \otimes \mathbb{C}^{(l)}\langle \underline{x} \rangle,$$

both $N + \mathrm{id}$ and $N \otimes \mathrm{id} + \mathrm{id} \otimes N + \mathrm{id} \otimes \mathrm{id}$ are diagonal and invertible.

Characterization of cyclic gradients I

Characterization of cyclic gradients I

Theorem (Voiculescu, 2000)

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. The following are equivalent:

1. $\underline{q}$ is a cyclic gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{q} = \mathcal{D}p$.

Characterization of cyclic gradients I

Theorem (Voiculescu, 2000)

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. The following are equivalent:

1. $\underline{q}$ is a cyclic gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{q} = \mathcal{D}p$.
2. $\displaystyle\sum_{j=1}^{n} [q_j, x_j] = 0$.

Characterization of cyclic gradients I

Theorem (Voiculescu, 2000)

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. The following are equivalent:

1. $\underline{q}$ is a cyclic gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{q} = \mathcal{D}p$.
2. $\displaystyle\sum_{j=1}^{n} [q_j, x_j] = 0$.
3. $\displaystyle\sum_{j=1}^{n} x_j q_j \in \operatorname{ran} C$.

Characterization of cyclic gradients I

Theorem (Voiculescu, 2000)

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. The following are equivalent:

1. $\underline{q}$ is a cyclic gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{q} = \mathcal{D}p$.
2. $\displaystyle\sum_{j=1}^n [q_j, x_j] = 0$.
3. $\displaystyle\sum_{j=1}^n x_j q_j \in \operatorname{ran} C$.
4. For $i = 1, \dots, n$, we have: $\displaystyle\mathcal{D}_i\Big(\sum_{j=1}^n x_j q_j\Big) = (N + \mathrm{id}) q_i$

Characterization of cyclic gradients I

Theorem (Voiculescu, 2000)

Let $\underline{q} = (q_1, \dots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ be given. The following are equivalent:

1. $\underline{q}$ is a cyclic gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{q} = \mathcal{D}p$.
2. $\displaystyle\sum_{j=1}^{n} [q_j, x_j] = 0$.
3. $\displaystyle\sum_{j=1}^{n} x_j q_j \in \operatorname{ran} C$.
4. For $i = 1, \dots, n$, we have: $\displaystyle\mathcal{D}_i\Big(\sum_{j=1}^{n} x_j q_j\Big) = (N + \mathrm{id}) q_i$

Integrability condition of Schwarz type (M., Speicher, 2019)

5. For $i, j = 1, \dots, n$, we have: $\partial_i q_j = \sigma(\partial_j q_i)$

Characterization of cyclic gradients II

Characterization of cyclic gradients II

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. We have checked that

$$\underline{q} = (q_1, q_2) := \big(2\, x_2 x_1 x_2,\ 2\, x_1 x_2 x_1\big)$$

is the cyclic gradient of p.

Characterization of cyclic gradients II

Example

Let $n = 2$ and consider again $p = x_1x_2x_1x_2$. We have checked that

$$\underline{q} = (q_1, q_2) := \big(2\,x_2x_1x_2,\ 2\,x_1x_2x_1\big)$$

is the cyclic gradient of p. We verify the *Schwarz integrability condition* by

$$\begin{aligned}
\partial_1 q_1 &= 2\,x_2 \otimes x_2,\\
\partial_2 q_1 &= 2\,(1 \otimes x_1x_2 + x_2x_1 \otimes 1),\\
\partial_1 q_2 &= 2\,(1 \otimes x_2x_1 + x_1x_2 \otimes 1),\\
\partial_2 q_2 &= 2\,x_1 \otimes x_1,
\end{aligned}$$

Characterization of cyclic gradients II

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. We have checked that

$$\underline{q} = (q_1, q_2) := \big(2\, x_2 x_1 x_2,\ 2\, x_1 x_2 x_1\big)$$

is the cyclic gradient of p. We verify the *Schwarz integrability condition* by

$$\begin{aligned}
\partial_1 q_1 &= 2\, x_2 \otimes x_2,\\
\partial_2 q_1 &= 2\,(1 \otimes x_1 x_2 + x_2 x_1 \otimes 1),\\
\partial_1 q_2 &= 2\,(1 \otimes x_2 x_1 + x_1 x_2 \otimes 1),\\
\partial_2 q_2 &= 2\, x_1 \otimes x_1,
\end{aligned}$$

from where we see that indeed

$$\partial_1 q_1 = \sigma(\partial_1 q_1), \quad \partial_2 q_2 = \sigma(\partial_2 q_2), \quad \text{and} \quad \partial_1 q_2 = \sigma(\partial_2 q_1).$$

How to find a cyclic antiderivative

How to find a cyclic antiderivative

Lemma

Suppose that $\underline{q} = (q_1, \ldots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ satisfies $\sum_{j=1}^n x_j q_j \in \operatorname{ran} C$. Then each $p \in \mathbb{C}\langle \underline{x} \rangle$ which solves the equation

$$Cp = \sum_{j=1}^n x_j q_j$$

is a cyclic antiderivative of $\underline{q}$, i.e., we have that $\mathcal{D}p = \underline{q}$.

How to find a cyclic antiderivative

Lemma

Suppose that $\underline{q} = (q_1, \ldots, q_n) \in \mathbb{C}\langle \underline{x} \rangle^n$ satisfies $\sum_{j=1}^n x_j q_j \in \operatorname{ran} C$. Then each $p \in \mathbb{C}\langle \underline{x} \rangle$ which solves the equation

$$Cp = \sum_{j=1}^{n} x_j q_j$$

is a cyclic antiderivative of $\underline{q}$, i.e., we have that $\mathcal{D}p = \underline{q}$.

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. For the tuple

$$\underline{q} = (q_1, q_2) = \big(2\, x_2 x_1 x_2,\ 2\, x_1 x_2 x_1\big),$$

of which we know that it is the cyclic derivative of p, we confirm that

$$x_1 q_1 + x_2 q_2 = 2\, x_1 x_2 x_1 x_2 + 2\, x_2 x_1 x_2 x_1 = C x_1 x_2 x_1 x_2 = Cp.$$

Characterization of free gradients I

Characterization of free gradients I

Theorem (M., Speicher, 2019)

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. Then the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{u} = \partial p$.

Characterization of free gradients I

Theorem (M., Speicher, 2019)

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. Then the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{u} = \partial p$.
2. There exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $$\sum_{j=1}^{n} u_j \sharp [x_j, 1 \otimes 1] = [p, 1 \otimes 1].$$

Characterization of free gradients I

Theorem (M., Speicher, 2019)

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. Then the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{u} = \partial p$.
2. There exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $$\sum_{j=1}^{n} u_j \sharp [x_j, 1 \otimes 1] = [p, 1 \otimes 1].$$
3. For $i, j = 1, \dots, n$, we have: $(\mathrm{id} \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id})(u_i)$

Characterization of free gradients I

Theorem (M., Speicher, 2019)

Let $\underline{u} = (u_1, \dots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. Then the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{u} = \partial p$.
2. There exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $$\sum_{j=1}^{n} u_j \sharp [x_j, 1 \otimes 1] = [p, 1 \otimes 1].$$
3. For $i, j = 1, \dots, n$, we have: $(\mathrm{id} \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id})(u_i)$
4. For $i = 1, \dots, n$: $\partial_i\Big(\sum_{j=1}^{n} u_j \sharp x_j\Big) = (N \otimes \mathrm{id} + \mathrm{id} \otimes N + \mathrm{id} \otimes \mathrm{id})(u_i)$

Characterization of free gradients I

Theorem (M., Speicher, 2019)

Let $\underline{u} = (u_1, \ldots, u_n) \in (\mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle)^n$ be given. Then the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $\underline{u} = \partial p$.
2. There exists $p \in \mathbb{C}\langle \underline{x} \rangle$ such that $$\sum_{j=1}^{n} u_j \sharp [x_j, 1 \otimes 1] = [p, 1 \otimes 1].$$
3. For $i, j = 1, \ldots, n$, we have: $(\mathrm{id} \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id})(u_i)$
4. For $i = 1, \ldots, n$: $$\partial_i \Big(\sum_{j=1}^{n} u_j \sharp x_j \Big) = (N \otimes \mathrm{id} + \mathrm{id} \otimes N + \mathrm{id} \otimes \mathrm{id})(u_i)$$

Every $p \in \mathbb{C}\langle \underline{x} \rangle$ which solves the equation $Np = \sum_{j=1}^{n} u_j \sharp x_j$ is a free antiderivative of $\underline{u}$, i.e., it satisfies $\underline{u} = \partial p$.

Characterization of free gradients II

Characterization of free gradients II

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. We have already seen that

$$\underline{u} = (u_1, u_2) := \big(1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,\ x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1\big)$$

is the free gradient of p.

Characterization of free gradients II

Example

Let $n = 2$ and consider again $p = x_1 x_2 x_1 x_2$. We have already seen that

$$\underline{u} = (u_1, u_2) := \big(1 \otimes x_2 x_1 x_2 + x_1 x_2 \otimes x_2,\ x_1 \otimes x_1 x_2 + x_1 x_2 x_1 \otimes 1\big)$$

is the free gradient of p. We verify the *Schwarz integrability condition* by

$$\begin{aligned}
(\partial_1 \otimes \mathrm{id}) u_1 &= 1 \otimes x_2 \otimes x_2 &&= (\mathrm{id} \otimes \partial_1) u_1 \\
(\partial_1 \otimes \mathrm{id}) u_2 &= 1 \otimes 1 \otimes x_1 x_2 + 1 \otimes x_2 x_1 \otimes 1 + x_1 x_2 \otimes 1 \otimes 1 &&= (\mathrm{id} \otimes \partial_2) u_1 \\
(\partial_2 \otimes \mathrm{id}) u_1 &= x_1 \otimes 1 \otimes x_2 &&= (\mathrm{id} \otimes \partial_1) u_2 \\
(\partial_2 \otimes \mathrm{id}) u_2 &= x_1 \otimes x_1 \otimes 1 &&= (\mathrm{id} \otimes \partial_2) u_2
\end{aligned}$$

Characterization of free gradients II

Example

Let $n = 2$ and consider again $p = x_1x_2x_1x_2$. We have already seen that

$$\underline{u} = (u_1, u_2) := \big(1 \otimes x_2x_1x_2 + x_1x_2 \otimes x_2,\ x_1 \otimes x_1x_2 + x_1x_2x_1 \otimes 1\big)$$

is the free gradient of p. We verify the *Schwarz integrability condition* by

$$\begin{aligned}
(\partial_1 \otimes \mathrm{id})u_1 &= 1 \otimes x_2 \otimes x_2 &&= (\mathrm{id} \otimes \partial_1)u_1 \\
(\partial_1 \otimes \mathrm{id})u_2 &= 1 \otimes 1 \otimes x_1x_2 + 1 \otimes x_2x_1 \otimes 1 + x_1x_2 \otimes 1 \otimes 1 &&= (\mathrm{id} \otimes \partial_2)u_1 \\
(\partial_2 \otimes \mathrm{id})u_1 &= x_1 \otimes 1 \otimes x_2 &&= (\mathrm{id} \otimes \partial_1)u_2 \\
(\partial_2 \otimes \mathrm{id})u_2 &= x_1 \otimes x_1 \otimes 1 &&= (\mathrm{id} \otimes \partial_2)u_2
\end{aligned}$$

We compute

$$u_1 \sharp x_1 + u_2 \sharp x_2 = 4\, x_1x_2x_1x_2 = N(x_1x_2x_1x_2) = Np.$$

Uncovering the underlying mechanism

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?
- Can one go beyond the polynomial case?

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?
- Can one go beyond the polynomial case?
- What role do the maps $\underline{u} \mapsto \sum_{j=1}^n u_j \sharp x_j$ and $\underline{q} \mapsto \sum_{j=1}^n x_j q_j$ play?

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?
- Can one go beyond the polynomial case?
- What role do the maps $\underline{u} \mapsto \sum_{j=1}^{n} u_j \sharp x_j$ and $\underline{q} \mapsto \sum_{j=1}^{n} x_j q_j$ play?

☞ **Voiculescu's multivariate generalized difference quotient rings!**

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle\underline{x}\rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?
- Can one go beyond the polynomial case?
- What role do the maps $\underline{u} \mapsto \sum_{j=1}^n u_j \sharp x_j$ and $\underline{q} \mapsto \sum_{j=1}^n x_j q_j$ play?

☞ **Voiculescu's multivariate generalized difference quotient rings!**

Free probability motivation

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space such that $\mathcal{M} = W^*(\underline{X})$ for some $\underline{X} = (X_1, \dots, X_n) \in \mathcal{M}_{\mathrm{sa}}^n$ with $\Phi^*(\underline{X}) < \infty$. It is known that the conjugate variables $\underline{\xi} = (\xi_1, \dots, \xi_n)$ of $\underline{X}$ have the following properties:

- $\sum_{j=1}^n [\xi_j, X_j] = 0$ [Voiculescu, 1999]

Uncovering the underlying mechanism

Questions

- How much do the results depend on the particular structure of $\mathbb{C}\langle \underline{x} \rangle$?
- Can the proofs be given by using only the arithmetic properties of free and cyclic derivatives?
- Can one go beyond the polynomial case?
- What role do the maps $\underline{u} \mapsto \sum_{j=1}^n u_j \sharp x_j$ and $\underline{q} \mapsto \sum_{j=1}^n x_j q_j$ play?

☞ **Voiculescu's multivariate generalized difference quotient rings!**

Free probability motivation

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space such that $\mathcal{M} = W^*(\underline{X})$ for some $\underline{X} = (X_1, \dots, X_n) \in \mathcal{M}_{\mathrm{sa}}^n$ with $\Phi^*(\underline{X}) < \infty$. It is known that the conjugate variables $\underline{\xi} = (\xi_1, \dots, \xi_n)$ of $\underline{X}$ have the following properties:

- $\sum_{j=1}^n [\xi_j, X_j] = 0$ [Voiculescu, 1999]
- $\overline{\partial}_{X_i} \xi_j = \sigma(\overline{\partial}_{X_j} \xi_i)$ if $\xi_1, \dots, \xi_n \in \bigcap_{j=1}^n \mathrm{dom}(\overline{\partial}_{X_j})$ [Dabrowski, 2014]

Voiculescu's multivariable GDQ rings

Voiculescu's multivariable GDQ rings

Definition (Voiculescu, 2000)

A multivariable generalized difference quotient ring (A, μ, ∂) consists of

- a complex (not necessarily unital) algebra A with the induced multiplication mapping $\mu : A \otimes A \to A, a_1 \otimes a_2 \mapsto a_1 a_2$, and
- a linear map $\partial = (\partial_1, \dots, \partial_n) : A \to (A \otimes A)^n$, the gradient of A,

such that the following conditions are satisfied:

Voiculescu's multivariable GDQ rings

Definition (Voiculescu, 2000)

A multivariable generalized difference quotient ring (A, μ, ∂) consists of

- a complex (not necessarily unital) algebra A with the induced multiplication mapping $\mu : A \otimes A \to A, a_1 \otimes a_2 \mapsto a_1 a_2$, and
- a linear map $\partial = (\partial_1, \dots, \partial_n) : A \to (A \otimes A)^n$, the gradient of A,

such that the following conditions are satisfied:

1. The mappings $\partial_1, \dots, \partial_n$ satisfy the joint coassociativity relation

$$(\partial_i \otimes \mathrm{id}_A) \circ \partial_j = (\mathrm{id}_A \otimes \partial_j) \circ \partial_i \qquad \text{for } i, j = 1, \dots, n.$$

Voiculescu's multivariable GDQ rings

Definition (Voiculescu, 2000)

A multivariable generalized difference quotient ring (A, μ, ∂) consists of

- a complex (not necessarily unital) algebra A with the induced multiplication mapping $\mu : A \otimes A \to A, a_1 \otimes a_2 \mapsto a_1 a_2$, and
- a linear map $\partial = (\partial_1, \ldots, \partial_n) : A \to (A \otimes A)^n$, the gradient of A,

such that the following conditions are satisfied:

1. The mappings $\partial_1, \ldots, \partial_n$ satisfy the joint coassociativity relation
$$(\partial_i \otimes \mathrm{id}_A) \circ \partial_j = (\mathrm{id}_A \otimes \partial_j) \circ \partial_i \qquad \text{for } i, j = 1, \ldots, n.$$
2. Each ∂_i is a $A \otimes A$-valued derivation on (A, μ), i.e., we have that
$$\partial_i \circ \mu = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_i) + (\mathrm{id}_A \otimes \mu) \circ (\partial_i \otimes \mathrm{id}_A).$$

Graded and weakly graded multivariable GDQ rings

Graded and weakly graded multivariable GDQ rings

Definition (Voiculescu, 2000; M., Speicher, 2019)

A multivariable GDQ ring (A, μ, ∂) is called

- weakly graded, if there exits a linear mapping $L : A \to A$ which is a coderivation with respect to each ∂_i, i.e., we have that

$$\partial_i \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_i.$$

  In this case, we say that L is a weak grading of (A, μ, ∂) and we call $N := L - \mathrm{id}_A$ the associated number operator.

Graded and weakly graded multivariable GDQ rings

Definition (Voiculescu, 2000; M., Speicher, 2019)

A multivariable GDQ ring (A, μ, ∂) is called

- weakly graded, if there exits a linear mapping $L : A \to A$ which is a coderivation with respect to each ∂_i, i.e., we have that

  $$\partial_i \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_i.$$

  In this case, we say that L is a weak grading of (A, μ, ∂) and we call $N := L - \mathrm{id}_A$ the associated number operator.

- graded, if (A, μ, ∂) admits a weak grading $L : A \to A$ for which the number operator N is an A-valued derivation on (A, μ), i.e.,

  $$N \circ \mu = \mu \circ (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N).$$

  In this case, we call L a grading of (A, μ, ∂).

Divergence for multivariable GDQ rings

Divergence for multivariable GDQ rings

Definition (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$ viewed as $\partial : A \to (A \otimes A)^n$. A divergence for (A, μ, ∂) is a linear map

$$\partial^\star := (\partial_1^\star, \dots, \partial_n^\star) : \ (A \otimes A)^n \to A, \quad \underline{u} = (u_1, \dots, u_n) \mapsto \sum_{j=1}^n \partial_j^\star(u_j)$$

such that for $i, j = 1, \dots, n$

$$\partial_j \circ \partial_i^\star = (\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) + \delta_{i,j} \, \mathrm{id}_{A \otimes A}.$$

Divergence for multivariable GDQ rings

Definition (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$ viewed as $\partial : A \to (A \otimes A)^n$. A divergence for (A, μ, ∂) is a linear map

$$\partial^\star := (\partial_1^\star, \dots, \partial_n^\star) : \ (A \otimes A)^n \to A, \quad \underline{u} = (u_1, \dots, u_n) \mapsto \sum_{j=1}^{n} \partial_j^\star(u_j)$$

such that for $i, j = 1, \dots, n$

$$\partial_j \circ \partial_i^\star = (\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) + \delta_{i,j} \, \mathrm{id}_{A \otimes A}.$$

Example

For the multivariable GDQ ring $(\mathbb{C}\langle \underline{x} \rangle, \mu, \partial)$ with the free gradient ∂, we get a divergence by

$$\partial_j^\star : \ \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad u \mapsto u \sharp x_j.$$

The induced (weak) grading

The induced (weak) grading

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$. Suppose that $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ is a divergence for (A, μ, ∂). Define

$$N: \ A \to A, \qquad N := \partial^\star \circ \partial = \sum_{j=1}^{n} \partial_j^\star \circ \partial_j$$

and $L := N + \mathrm{id}_A$. Then the following statements hold:

The induced (weak) grading

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$. Suppose that $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ is a divergence for (A, μ, ∂). Define

$$N:\ A \to A, \qquad N := \partial^\star \circ \partial = \sum_{j=1}^{n} \partial_j^\star \circ \partial_j$$

and $L := N + \mathrm{id}_A$. Then the following statements hold:

1. L is a weak grading on (A, μ, ∂).

The induced (weak) grading

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$. Suppose that $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ is a divergence for (A, μ, ∂). Define

$$N: \ A \to A, \qquad N := \partial^\star \circ \partial = \sum_{j=1}^{n} \partial_j^\star \circ \partial_j$$

and $L := N + \mathrm{id}_A$. Then the following statements hold:

1. L is a weak grading on (A, μ, ∂).
2. If each $\partial_j^\star$ is an A-bimodule homomorphism, then L is a grading.

The induced (weak) grading

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$. Suppose that $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ is a divergence for (A, μ, ∂). Define

$$N : \ A \to A, \qquad N := \partial^\star \circ \partial = \sum_{j=1}^{n} \partial_j^\star \circ \partial_j$$

and $L := N + \mathrm{id}_A$. Then the following statements hold:

1. L is a weak grading on (A, μ, ∂).
2. If each $\partial_j^\star$ is an A-bimodule homomorphism, then L is a grading.

Example

For the multivariable GDQ ring $(\mathbb{C}\langle \underline{x} \rangle, \mu, \partial)$ endowed with the divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ defined by $\partial^\star(u) = u \sharp x_j$, we get a grading $L = N + \mathrm{id}$ where $N = \partial^\star \circ \partial$ satisfies $N x_{i_1} x_{i_2} \cdots x_{i_k} = k\, x_{i_1} x_{i_2} \cdots x_{i_k}$.

Proof.

□

Proof.

We show that $L = N + \mathrm{id}_A$ with $N = \partial^\star \circ \partial = \sum_{i=1}^n \partial_i^\star \circ \partial_i$ is a weak grading, i.e.,

$$\partial_j \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

□

Proof.

We show that $L = N + \mathrm{id}_A$ with $N = \partial^\star \circ \partial = \sum_{i=1}^n \partial_i^\star \circ \partial_i$ is a weak grading, i.e.,

$$\partial_j \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_j \qquad \text{for } j = 1, \ldots, n.$$

We verify the latter identity as follows:

$$\partial_j \circ N = \sum_{i=1}^n (\partial_j \circ \partial_i^\star) \circ \partial_i$$

□

Proof.

We show that $L = N + \mathrm{id}_A$ with $N = \partial^\star \circ \partial = \sum_{i=1}^n \partial_i^\star \circ \partial_i$ is a weak grading, i.e.,

$$\partial_j \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

We verify the latter identity as follows:

$$\begin{aligned}\partial_j \circ N &= \sum_{i=1}^n (\partial_j \circ \partial_i^\star) \circ \partial_i \\ &\overset{(1)}{=} \partial_j + \sum_{i=1}^n \big((\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) \circ \partial_i + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) \circ \partial_i\big)\end{aligned}$$

where we use that

(1) $\partial_j \circ \partial_i^\star = (\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) + \delta_{i,j}\, \mathrm{id}_{A \otimes A}$

□

Proof.

We show that $L = N + \mathrm{id}_A$ with $N = \partial^\star \circ \partial = \sum_{i=1}^n \partial_i^\star \circ \partial_i$ is a weak grading, i.e.,

$$\partial_j \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

We verify the latter identity as follows:

$$\begin{aligned}
\partial_j \circ N &= \sum_{i=1}^n (\partial_j \circ \partial_i^\star) \circ \partial_i \\
&\overset{(1)}{=} \partial_j + \sum_{i=1}^n \big((\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) \circ \partial_i + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) \circ \partial_i \big) \\
&\overset{(2)}{=} \partial_j + \sum_{i=1}^n \big((\partial_i^\star \otimes \mathrm{id}_A) \circ (\partial_i \otimes \mathrm{id}_A) \circ \partial_j + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\mathrm{id}_A \otimes \partial_i) \circ \partial_j \big)
\end{aligned}$$

where we use that

(1) $\partial_j \circ \partial_i^\star = (\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) + \delta_{i,j} \, \mathrm{id}_{A \otimes A}$

(2) $(\mathrm{id}_A \otimes \partial_j) \circ \partial_i = (\partial_i \otimes \mathrm{id}_A) \circ \partial_j$ and $(\partial_j \otimes \mathrm{id}_A) \circ \partial_i = (\mathrm{id}_A \otimes \partial_i) \circ \partial_j$.

□

Proof.

We show that $L = N + \mathrm{id}_A$ with $N = \partial^\star \circ \partial = \sum_{i=1}^n \partial_i^\star \circ \partial_i$ is a weak grading, i.e.,

$$\partial_j \circ L = (L \otimes \mathrm{id}_A + \mathrm{id}_A \otimes L) \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

We verify the latter identity as follows:

$$\begin{aligned}
\partial_j \circ N &= \sum_{i=1}^n (\partial_j \circ \partial_i^\star) \circ \partial_i \\
&\overset{(1)}{=} \partial_j + \sum_{i=1}^n \big((\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) \circ \partial_i + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) \circ \partial_i \big) \\
&\overset{(2)}{=} \partial_j + \sum_{i=1}^n \big((\partial_i^\star \otimes \mathrm{id}_A) \circ (\partial_i \otimes \mathrm{id}_A) \circ \partial_j + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\mathrm{id}_A \otimes \partial_i) \circ \partial_j \big) \\
&= (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}) \circ \partial_j,
\end{aligned}$$

where we use that

(1) $\partial_j \circ \partial_i^\star = (\partial_i^\star \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \partial_j) + (\mathrm{id}_A \otimes \partial_i^\star) \circ (\partial_j \otimes \mathrm{id}_A) + \delta_{i,j} \, \mathrm{id}_{A \otimes A}$

(2) $(\mathrm{id}_A \otimes \partial_j) \circ \partial_i = (\partial_i \otimes \mathrm{id}_A) \circ \partial_j$ and $(\partial_j \otimes \mathrm{id}_A) \circ \partial_i = (\mathrm{id}_A \otimes \partial_i) \circ \partial_j$.

□

Characterization of free gradients

Characterization of free gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose that the following conditions are satisfied:

- There exists a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ for (A, μ, ∂); consider $N = \partial^\star \circ \partial$ and the weak grading $L = N + \mathrm{id}_A$.
- $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}$ is injective and $\mathrm{ran}\, \partial^\star \subseteq \mathrm{ran}\, N$.

Characterization of free gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose that the following conditions are satisfied:

- There exists a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ for (A, μ, ∂); consider $N = \partial^\star \circ \partial$ and the weak grading $L = N + \mathrm{id}_A$.
- $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}$ is injective and $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{u} = (u_1, \dots, u_n) \in (A \otimes A)^n$, the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $a \in A$ such that $\partial a = \underline{u}$.

Characterization of free gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose that the following conditions are satisfied:

- There exists a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ for (A, μ, ∂); consider $N = \partial^\star \circ \partial$ and the weak grading $L = N + \mathrm{id}_A$.
- $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}$ is injective and $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{u} = (u_1, \dots, u_n) \in (A \otimes A)^n$, the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $a \in A$ such that $\partial a = \underline{u}$.
2. For $i, j = 1, \dots, n$, we have: $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$

Characterization of free gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose that the following conditions are satisfied:

- There exists a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ for (A, μ, ∂); consider $N = \partial^\star \circ \partial$ and the weak grading $L = N + \mathrm{id}_A$.
- $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}$ is injective and $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{u} = (u_1, \dots, u_n) \in (A \otimes A)^n$, the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $a \in A$ such that $\partial a = \underline{u}$.
2. For $i, j = 1, \dots, n$, we have: $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$
3. For $j = 1, \dots, n$: $\partial_j\big(\partial^\star \underline{u}\big) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}) u_j$

Characterization of free gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose that the following conditions are satisfied:

- There exists a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ for (A, μ, ∂); consider $N = \partial^\star \circ \partial$ and the weak grading $L = N + \mathrm{id}_A$.
- $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}$ is injective and $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{u} = (u_1, \dots, u_n) \in (A \otimes A)^n$, the following are equivalent:

1. $\underline{u}$ is a free gradient, i.e., there exists $a \in A$ such that $\partial a = \underline{u}$.
2. For $i, j = 1, \dots, n$, we have: $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$
3. For $j = 1, \dots, n$: $\partial_j(\partial^\star \underline{u}) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A \otimes A}) u_j$

If the equivalent conditions are satisfied for $\underline{u}$, then every solution $a \in A$ of $Na = \partial^\star \underline{u}$ is a free antiderivative of $\underline{u}$, i.e., we have that $\partial a = \underline{u}$.

Proof.

□

Proof.

- If $u_j = \partial_j a$, then by the coassociativity relation

$$(\mathrm{id}_A \otimes \partial_i)(u_j) = (\mathrm{id}_A \otimes \partial_i)(\partial_j a) = (\partial_j \otimes \mathrm{id}_A)(\partial_i a) = (\partial_j \otimes \mathrm{id}_A)(u_i).$$

□

Proof.

- If $u_j = \partial_j a$, then by the coassociativity relation

$$(\mathrm{id}_A \otimes \partial_i)(u_j) = (\mathrm{id}_A \otimes \partial_i)(\partial_j a) = (\partial_j \otimes \mathrm{id}_A)(\partial_i a) = (\partial_j \otimes \mathrm{id}_A)(u_i).$$

- That $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$ implies

$$\partial_j\big(\partial^\star u\big) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j$$

can be checked like in the previous proof.

□

Proof.

- If $u_j = \partial_j a$, then by the coassociativity relation

$$(\mathrm{id}_A \otimes \partial_i)(u_j) = (\mathrm{id}_A \otimes \partial_i)(\partial_j a) = (\partial_j \otimes \mathrm{id}_A)(\partial_i a) = (\partial_j \otimes \mathrm{id}_A)(u_i).$$

- That $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$ implies

$$\partial_j(\partial^\star u) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j$$

can be checked like in the previous proof.

- Since $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$ by assumption, we find an element $a \in A$ such that $Na = \partial^\star \underline{u}$.

□

Proof.

- If $u_j = \partial_j a$, then by the coassociativity relation

$$(\mathrm{id}_A \otimes \partial_i)(u_j) = (\mathrm{id}_A \otimes \partial_i)(\partial_j a) = (\partial_j \otimes \mathrm{id}_A)(\partial_i a) = (\partial_j \otimes \mathrm{id}_A)(u_i).$$

- That $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$ implies

$$\partial_j(\partial^\star u) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j$$

  can be checked like in the previous proof.

- Since $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$ by assumption, we find an element $a \in A$ such that $Na = \partial^\star \underline{u}$. Since L is a weak grading, we get

$$\begin{aligned}&(N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})(\partial_j a)\\ &\qquad = \partial_j(Na) = \partial_j(\partial^\star \underline{u}) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j.\end{aligned}$$

□

Proof.

- If $u_j = \partial_j a$, then by the coassociativity relation

$$(\mathrm{id}_A \otimes \partial_i)(u_j) = (\mathrm{id}_A \otimes \partial_i)(\partial_j a) = (\partial_j \otimes \mathrm{id}_A)(\partial_i a) = (\partial_j \otimes \mathrm{id}_A)(u_i).$$

- That $(\mathrm{id}_A \otimes \partial_i)(u_j) = (\partial_j \otimes \mathrm{id}_A)(u_i)$ implies

$$\partial_j(\partial^\star u) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j$$

can be checked like in the previous proof.

- Since $\operatorname{ran} \partial^\star \subseteq \operatorname{ran} N$ by assumption, we find an element $a \in A$ such that $Na = \partial^\star \underline{u}$. Since L is a weak grading, we get

$$\begin{aligned}&(N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})(\partial_j a)\\ &\qquad = \partial_j(Na) = \partial_j(\partial^\star \underline{u}) = (N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A})u_j.\end{aligned}$$

Since $N \otimes \mathrm{id}_A + \mathrm{id}_A \otimes N + \mathrm{id}_{A\otimes A}$ is injective, we infer $\partial_j a = u_j$.

□

The cyclic derivatives in multivariable GDQ rings

The cyclic derivatives in multivariable GDQ rings

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$.

The cyclic derivatives in multivariable GDQ rings

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \ldots, \partial_n)$.

Definition

The cyclic derivatives associated to ∂ are the linear maps

$$\mathcal{D}_j : \; A \to A, \qquad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j \qquad \text{for } j = 1, \ldots, n.$$

We call $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_n) : A \to A^n$ the cyclic gradient.

The cyclic derivatives in multivariable GDQ rings

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$.

Definition

The cyclic derivatives associated to ∂ are the linear maps

$$\mathcal{D}_j : A \to A, \qquad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

We call $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n) : A \to A^n$ the cyclic gradient.

Theorem (M., Speicher, 2019)

$$\sigma \circ \partial_i \circ \mathcal{D}_j = \partial_j \circ \mathcal{D}_i \qquad \text{for } i, j = 1, \dots, n.$$

The cyclic derivatives in multivariable GDQ rings

Let (A, μ, ∂) be a multivariable GDQ ring with gradient $\partial = (\partial_1, \dots, \partial_n)$.

Definition

The cyclic derivatives associated to ∂ are the linear maps

$$\mathcal{D}_j : \ A \to A, \qquad \mathcal{D}_j := \mu \circ \sigma \circ \partial_j \qquad \text{for } j = 1, \dots, n.$$

We call $\mathcal{D} = (\mathcal{D}_1, \dots, \mathcal{D}_n) : A \to A^n$ the cyclic gradient.

Theorem (M., Speicher, 2019)

$$\sigma \circ \partial_i \circ \mathcal{D}_j = \partial_j \circ \mathcal{D}_i \qquad \text{for } i, j = 1, \dots, n.$$

Theorem (M., Speicher, 2019)

Let $L : A \to A$ be a grading on (A, μ, ∂) and let $N = L - \mathrm{id}_A$ be the associated number operator. Then

$$\mathcal{D}_i \circ N = L \circ \mathcal{D}_i \qquad \text{for } i = 1, \dots, n.$$

Cyclic divergence

Cyclic divergence

Definition (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring and let $\partial^\star$ be a divergence. A cyclic divergence for (A, μ, ∂) (compatible with $\partial^\star$) is a linear map

$$\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star) : \; A^n \to A, \quad \underline{b} = (b_1, \dots, b_n) \mapsto \sum_{i=1}^n \mathcal{D}_i^\star(b_i)$$

such that for $i, j = 1, \dots, n$: $\quad \mathcal{D}_j \circ \mathcal{D}_i^\star = \partial_i^\star \circ \sigma \circ \partial_j + \delta_{i=j} \operatorname{id}_A$

Cyclic divergence

Definition (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring and let $\partial^\star$ be a divergence. A cyclic divergence for (A, μ, ∂) (compatible with $\partial^\star$) is a linear map

$$\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star) : \ A^n \to A, \quad \underline{b} = (b_1, \dots, b_n) \mapsto \sum_{i=1}^n \mathcal{D}_i^\star(b_i)$$

such that for $i, j = 1, \dots, n$: $\quad \mathcal{D}_j \circ \mathcal{D}_i^\star = \partial_i^\star \circ \sigma \circ \partial_j + \delta_{i=j} \operatorname{id}_A$

Example

If $(\mathbb{C}\langle \underline{x} \rangle, \mu, \partial)$ is endowed with the divergence $\partial_j^\star : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ defined by $\partial_j^\star(u) = u \sharp x_j$, then we get a compatible cyclic divergence by

$$\mathcal{D}_j^\star : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad b \mapsto x_j b.$$

Indeed: $\quad \mathcal{D}_j(\mathcal{D}_i^\star b) = \mathcal{D}_j(x_i b) = (\tilde{\partial}_j b) \sharp x_i + (\tilde{\partial}_j x_i) \sharp b = \partial_i^\star(\tilde{\partial}_j b) + \delta_{i=j} b$

Cyclic divergence

Definition (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring and let $\partial^\star$ be a divergence. A cyclic divergence for (A, μ, ∂) (compatible with $\partial^\star$) is a linear map

$$\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star) : \ A^n \to A, \quad \underline{b} = (b_1, \dots, b_n) \mapsto \sum_{i=1}^{n} \mathcal{D}_i^\star(b_i)$$

such that for $i, j = 1, \dots, n$: $\quad \mathcal{D}_j \circ \mathcal{D}_i^\star = \partial_i^\star \circ \sigma \circ \partial_j + \delta_{i=j} \operatorname{id}_A$

Example

If $(\mathbb{C}\langle \underline{x} \rangle, \mu, \partial)$ is endowed with the divergence $\partial_j^\star : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle$ defined by $\partial_j^\star(u) = u \sharp x_j$, then we get a compatible cyclic divergence by

$$\mathcal{D}_j^\star : \ \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, \quad b \mapsto x_j b = \partial_j^\star(1 \otimes b).$$

Indeed: $\quad \mathcal{D}_j(\mathcal{D}_i^\star b) = \mathcal{D}_j(x_i b) = (\tilde{\partial}_j b) \sharp x_i + (\tilde{\partial}_j x_i) \sharp b = \partial_i^\star(\tilde{\partial}_j b) + \delta_{i=j} b$

The associated cyclic symmetrization operator

The associated cyclic symmetrization operator

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring and let $\partial^\star$ be a divergence. To a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$, we associate the cyclic symmetrization operator

$$C: \ A \to A, \qquad C := \mathcal{D}^\star \circ \mathcal{D} = \sum_{i=1}^n \mathcal{D}_i^\star \circ \mathcal{D}_i.$$

Then, for $j = 1, \dots, n$: $\qquad \mathcal{D}_j \circ C = L \circ \mathcal{D}_j$

The associated cyclic symmetrization operator

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring and let $\partial^\star$ be a divergence. To a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$, we associate the cyclic symmetrization operator

$$C: \ A \to A, \qquad C := \mathcal{D}^\star \circ \mathcal{D} = \sum_{i=1}^{n} \mathcal{D}_i^\star \circ \mathcal{D}_i.$$

Then, for $j = 1, \dots, n$: $\qquad \mathcal{D}_j \circ C = L \circ \mathcal{D}_j$

Example

On $(\mathbb{C}\langle \underline{x} \rangle, \mu, \partial)$, we consider $\partial_j^\star : \mathbb{C}\langle \underline{x} \rangle \otimes \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, u \mapsto u \sharp x_j$ and $\mathcal{D}_j^\star : \mathbb{C}\langle \underline{x} \rangle \to \mathbb{C}\langle \underline{x} \rangle, b \mapsto x_j b$, then $C = \mathcal{D}^\star \circ \mathcal{D}$ satisfies

$$C x_{i_1} \cdots x_{i_k} = \sum_{p=1}^{k} x_{i_{p+1}} \cdots x_{i_k} x_{i_1} \cdots x_{i_p}.$$

Characterization of cyclic gradients

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran} \mathcal{D}^\star \subseteq \operatorname{ran} N$.

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran} \mathcal{D}^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{b} = (b_1, \dots, b_n) \in A^n$, the following are equivalent:

1. $\underline{b}$ is a cyclic gradient, i.e., there exists $a \in A$ such that $\mathcal{D}a = \underline{b}$.

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran} \mathcal{D}^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{b} = (b_1, \dots, b_n) \in A^n$, the following are equivalent:

1. $\underline{b}$ is a cyclic gradient, i.e., there exists $a \in A$ such that $\mathcal{D}a = \underline{b}$.
2. For $i, j = 1, \dots, n$, we have : $\partial_i b_j = \sigma\big(\partial_j b_i\big)$

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran} \mathcal{D}^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{b} = (b_1, \dots, b_n) \in A^n$, the following are equivalent:

1. $\underline{b}$ is a cyclic gradient, i.e., there exists $a \in A$ such that $\mathcal{D}a = \underline{b}$.
2. For $i, j = 1, \dots, n$, we have : $\quad \partial_i b_j = \sigma\big(\partial_j b_i\big)$
3. For $j = 1, \dots, n$, we have: $\quad \mathcal{D}_j\big(\mathcal{D}^\star \underline{b}\big) = L b_j$.

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A, μ, ∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star, \dots, \partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star, \dots, \mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran} \mathcal{D}^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{b} = (b_1, \dots, b_n) \in A^n$, the following are equivalent:

1. $\underline{b}$ is a cyclic gradient, i.e., there exists $a \in A$ such that $\mathcal{D}a = \underline{b}$.
2. For $i, j = 1, \dots, n$, we have : $\partial_i b_j = \sigma\big(\partial_j b_i\big)$
3. For $j = 1, \dots, n$, we have: $\mathcal{D}_j\big(\mathcal{D}^\star \underline{b}\big) = L b_j$.

If the equivalent conditions are satisfied for $\underline{b}$, then every $a \in A$ solving $Na = \partial^\star \underline{u}$ is a cyclic antiderivative of $\underline{b}$, i.e., we have that $\mathcal{D}a = \underline{b}$.

Characterization of cyclic gradients

Theorem (M., Speicher, 2019)

Let (A,μ,∂) be a multivariable GDQ ring. Suppose the following:

- There is a divergence $\partial^\star = (\partial_1^\star,\dots,\partial_n^\star)$ consisting of A-bimodule homomorphisms; put $N = \partial^\star \circ \partial$ and the grading $L = N + \mathrm{id}_A$.
- There is a cyclic divergence $\mathcal{D}^\star = (\mathcal{D}_1^\star,\dots,\mathcal{D}_n^\star)$ compatible with $\partial^\star$.
- The grading $L : A \to A$ is injective and it holds $\operatorname{ran}\mathcal{D}^\star \subseteq \operatorname{ran} N$.

Then, for any $\underline{b} = (b_1,\dots,b_n) \in A^n$, the following are equivalent:

1. $\underline{b}$ is a cyclic gradient, i.e., there exists $a \in A$ such that $\mathcal{D}a = \underline{b}$.
2. For $i,j = 1,\dots,n$, we have : $\partial_i b_j = \sigma\big(\partial_j b_i\big)$
3. For $j = 1,\dots,n$, we have: $\mathcal{D}_j\big(\mathcal{D}^\star \underline{b}\big) = L b_j$.

If the equivalent conditions are satisfied for $\underline{b}$, then every $a \in A$ solving $Na = \partial^\star \underline{u}$ is a cyclic antiderivative of $\underline{b}$, i.e., we have that $\mathcal{D}a = \underline{b}$. The same is true for each $a \in A$ solving $Ca = \partial^\star \underline{u}$, where $C = \mathcal{D}^\star \circ \mathcal{D}$ is the cyclic symmetrization operator.

Example: Chebyshev polynomials ($n = 1$)

Example: Chebyshev polynomials ($n = 1$)

We consider the linear bases of $\mathbb{C}\langle x\rangle$ given by the Chebyshev polynomials:

$$\begin{aligned} &\text{first kind}: && t_0 = 2, && t_1 = x, && x t_k = t_{k+1} + t_{k-1} && (k \geq 1) \\ &\text{second kind}: && u_0 = 1, && u_1 = x, && x u_k = u_{k+1} + u_{k-1} && (k \geq 1) \end{aligned}$$

Example: Chebyshev polynomials ($n = 1$)

We consider the linear bases of $\mathbb{C}\langle x \rangle$ given by the Chebyshev polynomials:

$$\begin{aligned} &\text{first kind}: && t_0 = 2, && t_1 = x, && x t_k = t_{k+1} + t_{k-1} && (k \geq 1) \\ &\text{second kind}: && u_0 = 1, && u_1 = x, && x u_k = u_{k+1} + u_{k-1} && (k \geq 1) \end{aligned}$$

- For $\partial : \mathbb{C}\langle x \rangle \to \mathbb{C}\langle x \rangle \otimes \mathbb{C}\langle x \rangle$ and $\mathcal{D} : \mathbb{C}\langle x \rangle \to \mathbb{C}\langle x \rangle$, we obtain

$$\partial u_k = \sum_{p=1}^{k} u_{p-1} \otimes u_{k-p} \qquad \text{and} \qquad \mathcal{D} t_{k+1} = (k+1) u_k.$$

Example: Chebyshev polynomials ($n = 1$)

We consider the linear bases of $\mathbb{C}\langle x\rangle$ given by the Chebyshev polynomials:

first kind :	$t_0 = 2,$	$t_1 = x,$	$xt_k = t_{k+1} + t_{k-1}$	$(k \geq 1)$
second kind :	$u_0 = 1,$	$u_1 = x,$	$xu_k = u_{k+1} + u_{k-1}$	$(k \geq 1)$

- For $\partial : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ and $\mathcal{D} : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$, we obtain

$$\partial u_k = \sum_{p=1}^{k} u_{p-1} \otimes u_{k-p} \qquad \text{and} \qquad \mathcal{D}t_{k+1} = (k+1)u_k.$$

- We define $\partial^\star : \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ and $\mathcal{D}^\star : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ by

$$\partial^\star(u_k \otimes u_l) := u_{k+l+1} \qquad \text{and} \qquad \mathcal{D}^\star u_k = t_{k+1}.$$

Example: Chebyshev polynomials ($n = 1$)

We consider the linear bases of $\mathbb{C}\langle x\rangle$ given by the Chebyshev polynomials:

first kind : $t_0 = 2,\quad t_1 = x,\quad x t_k = t_{k+1} + t_{k-1}\quad (k \geq 1)$

second kind : $u_0 = 1,\quad u_1 = x,\quad x u_k = u_{k+1} + u_{k-1}\quad (k \geq 1)$

- For $\partial : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ and $\mathcal{D} : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$, we obtain

$$\partial u_k = \sum_{p=1}^{k} u_{p-1} \otimes u_{k-p} \qquad \text{and} \qquad \mathcal{D} t_{k+1} = (k+1) u_k.$$

- We define $\partial^\star : \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ and $\mathcal{D}^\star : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ by

$$\partial^\star(u_k \otimes u_l) := u_{k+l+1} \qquad \text{and} \qquad \mathcal{D}^\star u_k = t_{k+1}.$$

- Thus, the number operators satisfy $N u_k = k\, u_k$ and $C t_k = k\, t_k$.

Example: Chebyshev polynomials ($n = 1$)

We consider the linear bases of $\mathbb{C}\langle x\rangle$ given by the Chebyshev polynomials:

$$\text{first kind}: \quad t_0 = 2, \quad t_1 = x, \quad xt_k = t_{k+1} + t_{k-1} \quad (k \geq 1)$$

$$\text{second kind}: \quad u_0 = 1, \quad u_1 = x, \quad xu_k = u_{k+1} + u_{k-1} \quad (k \geq 1)$$

- For $\partial : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle$ and $\mathcal{D} : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$, we obtain

$$\partial u_k = \sum_{p=1}^{k} u_{p-1} \otimes u_{k-p} \quad \text{and} \quad \mathcal{D} t_{k+1} = (k+1) u_k.$$

- We define $\partial^\star : \mathbb{C}\langle x\rangle \otimes \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ and $\mathcal{D}^\star : \mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle$ by

$$\partial^\star(u_k \otimes u_l) := u_{k+l+1} \quad \text{and} \quad \mathcal{D}^\star u_k = t_{k+1}.$$

- Thus, the number operators satisfy $N u_k = k\, u_k$ and $C t_k = k\, t_k$.

Thank you!