Regularity properties of limiting eigenvalue distributions by means of free probability theory

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Workshop "Sums & Products of Random Matrices"

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Random matrices: the general frame

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Definition (Random matrices)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Elements in the complex $*$-algebra

$$\mathfrak{A}_N := M_N(L^{\infty-}(\Omega, \mathbb{P})), \quad \text{where} \quad L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathbb{P}),$$

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are called random matrices (of size $N \times N$).

Remark

Each $X \in \mathfrak{A}_N$ is an $M_N(\mathbb{C})$-valued random variable $X : \Omega \to M_N(\mathbb{C})$ and thus induces a Borel probability measure Λ on $M_N(\mathbb{C})$ (or on $M_N(\mathbb{C})_{\mathrm{sa}}$, in case that X is selfadjoint) as the push-forward

$$\Lambda := X_*(\mathbb{P}).$$

Thus, random matrices of size $N \times N$ can alternatively be regarded as elements in a classical probability space $(M_N(\mathbb{C}), \Lambda)$ (or $(M_N(\mathbb{C})_{\mathrm{sa}}, \Lambda)$).

Random matrices: GUE's

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Definition (Gaussian random matrix)

A standard Gaussian random matrix (of size $N \times N$) is a selfadjoint random matrix $X = (X_{k,l})_{k,l=1}^N \in \mathfrak{A}_N$ for which

$$\{\mathrm{Re}(X_{k,l}) \mid 1 \leq k \leq l \leq N\} \cup \{\mathrm{Im}(X_{k,l}) \mid 1 \leq k < l \leq N\}$$

are independent Gaussian random variables such that

$$\mathbb{E}[X_{k,l}] = 0 \quad \text{and} \quad \mathbb{E}[|X_{k,l}|^2] = N^{-1} \quad \text{for } 1 \leq k \leq l \leq N.$$

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A standard Gaussian random matrix follows the law of the GUE, which is the probability measure Λ_N on $M_N(\mathbb{C})_{\mathrm{sa}} \cong \mathbb{R}^{N^2}$ that is determined by

$$d\Lambda_N(X) := \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{Tr}(X^2)}\, dX \quad \text{with} \quad Z_N := 2^{N/2}\Big(\frac{\pi}{N}\Big)^{N^2/2}$$

and $\quad dX := \prod_{k=1}^N dX_{k,k} \prod_{1\leq k<l\leq N} d\,\mathrm{Re}(X_{k,l})\, d\,\mathrm{Im}(X_{k,l}).$

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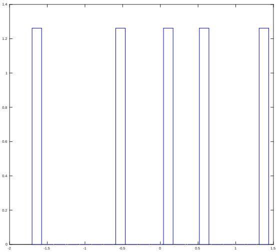

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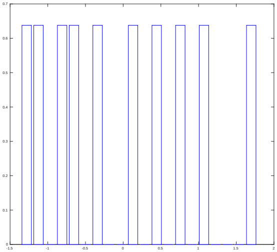

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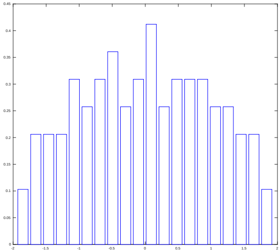

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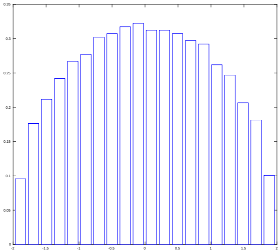

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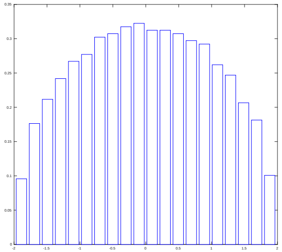

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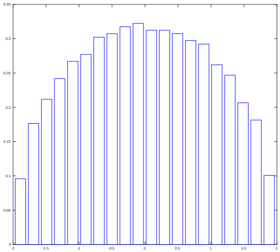

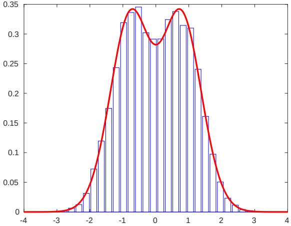

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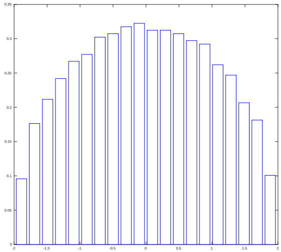

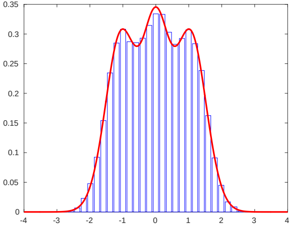

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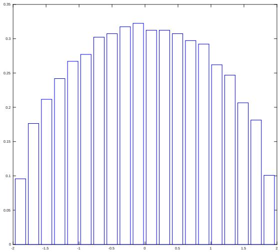

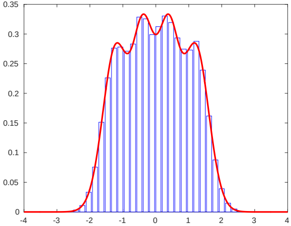

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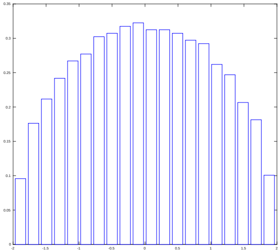

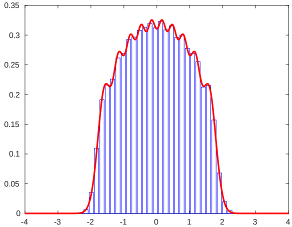

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Gaussian random matrices
$\rightsquigarrow$ Wigner's semicircle theorem

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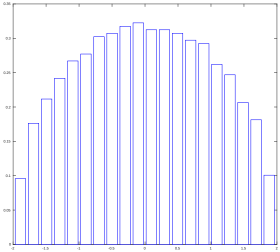

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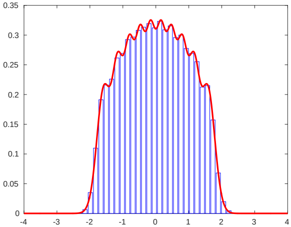

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For each $N \in \mathbb{N}$, let independent standard Gaussian random matrices $X_1^{(N)}, \dots, X_n^{(N)}$ be given. Further, take a noncommutative polynomial or rational function f. We combined

- the analytic subordination machinery of operator-valued free probability theory with
- algebraic linearization techniques
- and the hermitization method

in order to compute the (expected) limiting eigenvalue distribution of

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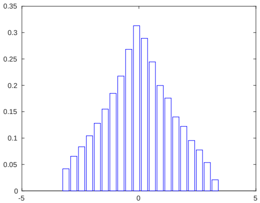

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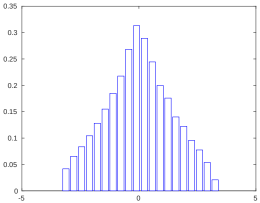

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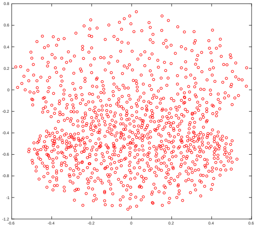

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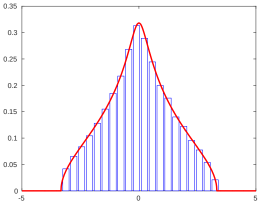

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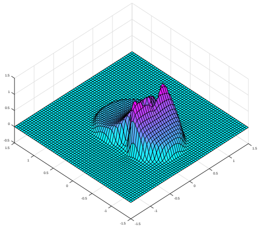

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$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

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Fact

A matrix $Q \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ is invertible in $M_N(\mathbb{C}(\!\langle x_1, \dots, x_n \rangle\!))$ if and only if it is full, i.e., if there is *no* $1 \leq k < N$ so that Q can be written in the form $Q = R_1 R_2$ for some rectangular matrices

$$R_1 \in M_{N \times k}(\mathbb{C}\langle x_1, \dots, x_n \rangle) \quad \text{and} \quad R_2 \in M_{k \times N}(\mathbb{C}\langle x_1, \dots, x_n \rangle).$$

Otherwise, we denote by $\operatorname{rank}(Q)$ the minimal k with this property.

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Definition ("analytic distribution")

Let $(\mathcal{M}, \tau)$ be a W^*-probability space. The (analytic) distribution of $X = X^* \in \mathcal{M}$ is the unique Borel probability measure μ_X on $\mathbb{R}$ that satisfies

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all } k = 0, 1, 2, \ldots.$$

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We have the following multivariate version of Wigner's semicircle law.

Theorem (Voiculescu (1991))

For all $N \in \mathbb{N}$, realize independent standard Gaussian random matrices $X_1^{(N)}, \dots, X_n^{(N)}$. Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)}))] = \tau(P(S_1, \dots, S_n))$$

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This means: Asymptotic freeness relates

- the limiting eigenvalue distribution of $Y^{(N)} = P(X_1^{(N)}, \dots, X_n^{(N)})$ and
- the distribution of $Y = P(S_1, \dots, S_n)$ for freely independent semicircular elements $S_1, \dots, S_n$.

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- $X_1^{(N)}, \dots, X_n^{(N)}$, independent standard Gaussian random matrices
- f, nc polynomial or nc rational function

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What other random matrix models?

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Definition (Gibbs laws)

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Asymptotic freeness revisited

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Let $V \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ with law Λ_N^V. Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$,

$$\lim_{N\to\infty} \mathrm{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)})) = \tau(P(X_1, \dots, X_n)) \quad \text{almost surely}$$

for selfadjoint operators $X_1, \dots, X_n$ in some W^*-probability space $(\mathcal{M}, \tau)$ that satisfy the Schwinger-Dyson equation, i.e.

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This means: We have a relation between

- the limiting eigenvalue distribution of $Y^{(N)} = P(X_1^{(N)}, \dots, X_n^{(N)})$ and
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Noncommutative and cyclic derivatives

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Definition

(i) The noncommutative derivatives are the linear mappings

$$\partial_1, \ldots, \partial_n : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}\langle x_1, \ldots, x_n \rangle \otimes \mathbb{C}\langle x_1, \ldots, x_n \rangle$$

which are uniquely determined by the two conditions

- $\partial_j(P_1 P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2)$ for all $P_1, P_2 \in \mathbb{C}\langle z_1, \ldots, z_n \rangle$,
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$$D_1, \dots, D_n : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle$$

that are defined by $D_j := \tilde{m} \circ \partial_j$, where

$$\tilde{m} : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

denotes the flipped multiplication defined as $\tilde{m}(P_1 \otimes P_2) := P_2 P_1$.

Conjugate variables and free Fisher information

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Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and consider any selfadjoint noncommutative random variables $X_1, \ldots, X_n \in \mathcal{M}$.

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Definition (Voiculescu (1998); M., Speicher, Weber (2014))

If $\xi_1, \dots, \xi_n \in L^2(X_1, \dots, X_n; \tau)$ are such that for all $P \in \mathbb{C}\langle z_1, \dots, z_n \rangle$

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)), \quad j = 1, \dots, n,$$

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The (non-microstates) free Fisher information is defined by

$$\Phi^*(X_1,\dots,X_n) := \begin{cases} \sum_{j=1}^n \|\xi_j\|_2^2, & \text{if a conjugate system } (\xi_1,\dots,\xi_n) \text{ for } (X_1,\dots,X_n) \text{ exists} \\ \infty, & \text{otherwise} \end{cases}$$

A useful variant of free entropy dimension

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Suppose that $S_1, \dots, S_n$ are freely independent semicircular elements that are also free from $\{X_1, \dots, X_n\}$, then $(X_1 + \sqrt{t}S_n, \dots, X_n + \sqrt{t}S_n)$ admits a conjugate system for each $t > 0$. More precisely, we have

$$\frac{n^2}{C^2 + nt} \leq \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n) \leq \frac{n}{t} \quad \text{for all } t > 0,$$

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Philosophy

If $\delta^\star(X_1, \dots, X_n) = n$, then $(X_1, \dots, X_n)$ has *no atomic part*.

Absence of atoms

Suppose that $\delta^\star(X_1, \ldots, X_n) = n$.

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Theorem (CS16, MSW17)

Let $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ be selfadjoint and non-constant. Then $\mu_{P(X_1,\ldots,X_n)}$ has no atoms.

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Theorem (MSY18)

Take selfadjoint matrices $b_0,b_1,\dots,b_n\in M_N(\mathbb{C})$, then the matrix-valued element $\mathbf{Y}:=\mathbf{P}(X_1,\dots,X_n)$ for

$$\mathbf{P}:=b_0+b_1x_1+\dots+b_nx_n\in M_N(\mathbb{C}\langle x_1,\dots,x_n\rangle)$$

has atoms precisely at the points in the set

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free analysis ⇣ (from CS16, MSW17 down to the theorem below) ↑ (from the theorem below up to MSY18)

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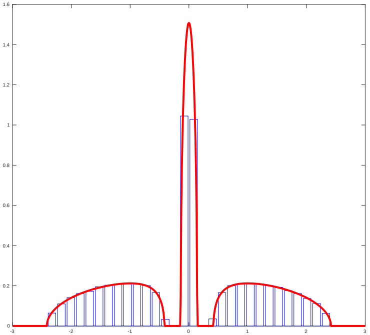
$N = 1000$, i.e.,
$\mathbf{X}^{(N)}$ of size 3000×3000.



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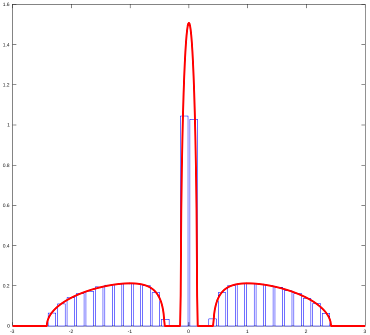
[Ajanki, Erdös, Krüger (2016)]
[Alt, Erdös, Krüger (2018)]



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Let $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be selfadjoint with degree $d \geq 1$ and consider

$$Y := P(X_1, \dots, X_n).$$

Then there exists some constant $C > 0$ (depending on P and $X_1, \dots, X_n$) such that the cumulative distribution function $\mathcal{F}_Y$ of Y, which is defined as $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$, satisfies

$$|\mathcal{F}_Y(t) - \mathcal{F}_Y(s)| \leq C|t-s|^{\frac{2}{2^{d+2}-5}} \qquad \text{for all } s, t \in \mathbb{R}.$$

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In fact, for every $R > \max_{i=1,\dots,n} \|X_i\|$, we can take

$$C = \prod_{k=1}^{d-1} \Big(\frac{d!}{(d-k)!}\Big)^{\frac{2^k}{2^{d+2}-5}} \rho_R(P)^{-\frac{2^d}{2^{d+2}-5}} \|P\|_R^{-\frac{2}{2^{d+2}-5}} \big(8R\Phi^*(X)^{1/2}\big)^{\frac{2(2^d-1)}{2^{d+2}-5}}.$$

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Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be selfadjoint with degree $d \geq 1$; consider

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Then, the logarithmic energy (and thus also the free entropy $\chi^*(Y)$)

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Remark

This is a first step towards a conjecture of Charlesworth and Shlyakhtenko saying that this should hold under the weaker condition

$$\chi^*(X_1, \ldots, X_n) > -\infty.$$

Hölder continuity – consequences II

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Corollary (Banna, M. (2018))

Let $V \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ be "nice" and let $(X_1^{(N)}, \dots, X_n^{(N)})$ be random matrices of size $N \times N$ distributed according to the Gibbs law

$$d\Lambda_N^V(X_1, \dots, X_n) = \frac{1}{Z_N^V} e^{-N \operatorname{Tr}(V(X_1, \dots, X_n))} \, dX_1 \, \dots \, dX_n.$$

Hölder continuity – consequences II

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Then, for each non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, we have that

(i) The empirical eigenvalue distribution $\mu_{Y^{(N)}}$ of

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converges in distribution almost surely to a compactly supported Borel probability measure μ on $\mathbb{R}$ with a cumulative distribution function that is Hölder continuous with exponent $\frac{2}{2^{d+2}-5}$.

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Hölder continuity – consequences II

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Corollary (M., Speicher, Yin (2018); Banna, M. (2018))

Let $b_0, b_1, \ldots, b_n \in M_d(\mathbb{C})$ be selfadjoint such that the quantum operator

$$\mathcal{L}: \ M_d(\mathbb{C}) \to M_d(\mathbb{C}), \quad b \mapsto b_1 b b_1 + \cdots + b_n b b_n$$

satisfies $\mathcal{L}(b) \geq c \operatorname{tr}_d(b) 1_d$ for all positive $b \in M_d(\mathbb{C})$ for some $c > 0$. Put

$$\mathbf{X}^{(N)} := b_0 \otimes 1_N + b_1 \otimes X_1^{(N)} + \cdots + b_n \otimes X_n^{(N)}$$

for independent standard Gaussian random matrices $(X_1^{(N)}, \ldots, X_n^{(N)})$ and

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(i) $\mathcal{F}_\mathbf{S}$ is Hölder continuous with exponent $\frac{2}{3}$.

(ii) There is some $C > 0$ such that $\Delta(\overline{\mu}_{\mathbf{X}^{(N)}}, \mu_\mathbf{S}) \leq C N^{-4/29}$.

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Thank you!